Learning quantum states efficiently

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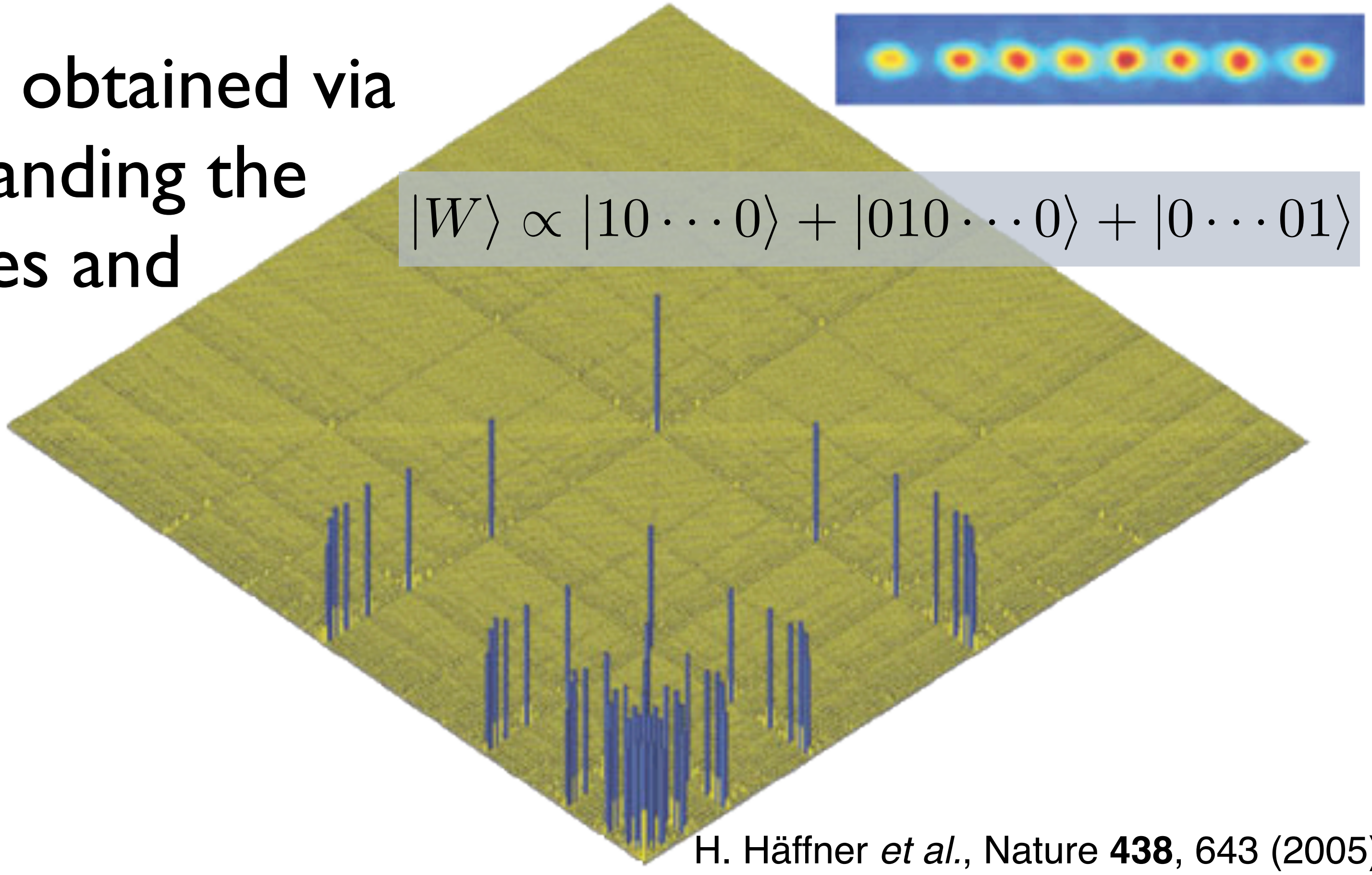
Institut für Theoretische Physik, Universität Ulm

- Introduction
    - A few experimental examples
    - Learning vs. verifying
    - The problem
- Some schemes
    - Permutationally invariant tomography
    - Evidence procedure
    - Compressed sensing
- Efficient state representation
    - and using it for tomography

State tomography

"Full information of the *N*-ion state is obtained via quantum state reconstruction by expanding the density matrix in a basis of observables and measuring the corresponding expectation values."

$$\hat{\varrho} = \begin{pmatrix} \varrho_{1,1} & \cdots & \varrho_{1,d} \\ \vdots & \ddots & \vdots \\ \varrho_{d,1} & \cdots & \varrho_{d,d} \end{pmatrix}$$



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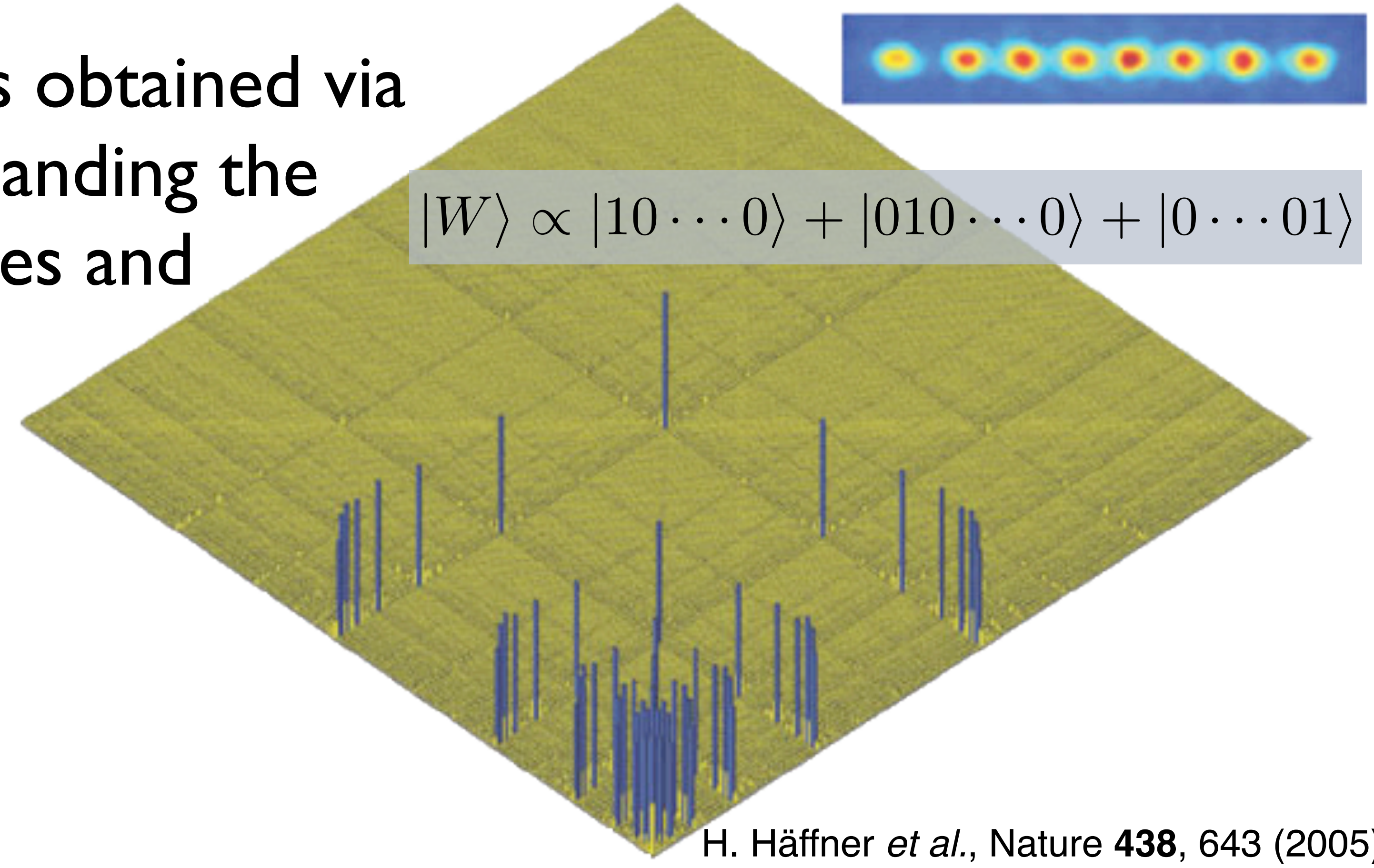

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$$= \sum_{\delta,\delta'=1}^{d} \mathrm{tr}[\hat{e}^{\dagger}_{\delta,\delta'}\hat{\varrho}]\hat{e}_{\delta,\delta'} = \sum_{\delta=1}^{d^2} \boxed{\mathrm{tr}[\hat{B}^{\dagger}_{\delta}\hat{\varrho}]}\hat{B}_{\delta}$$

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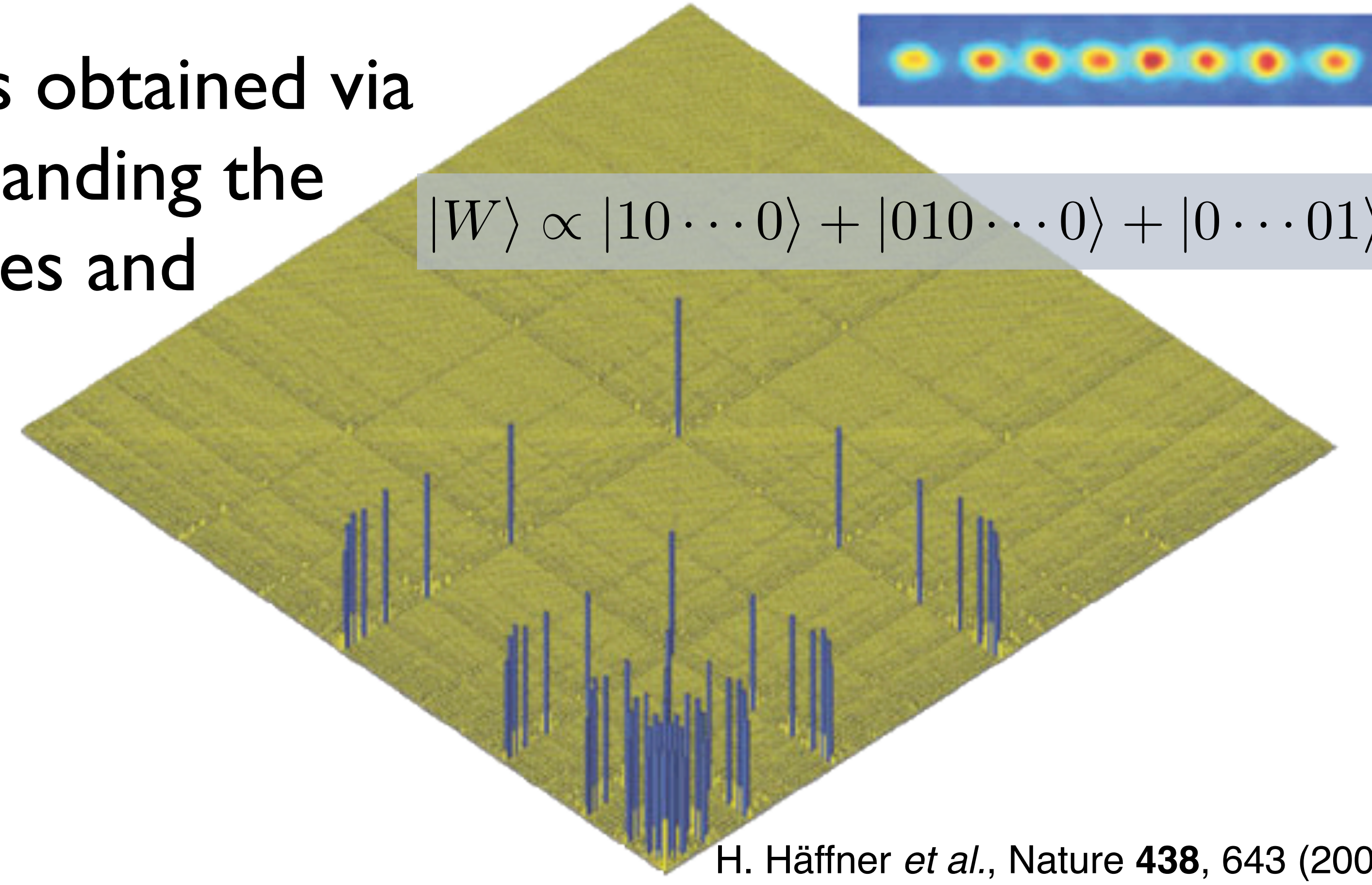

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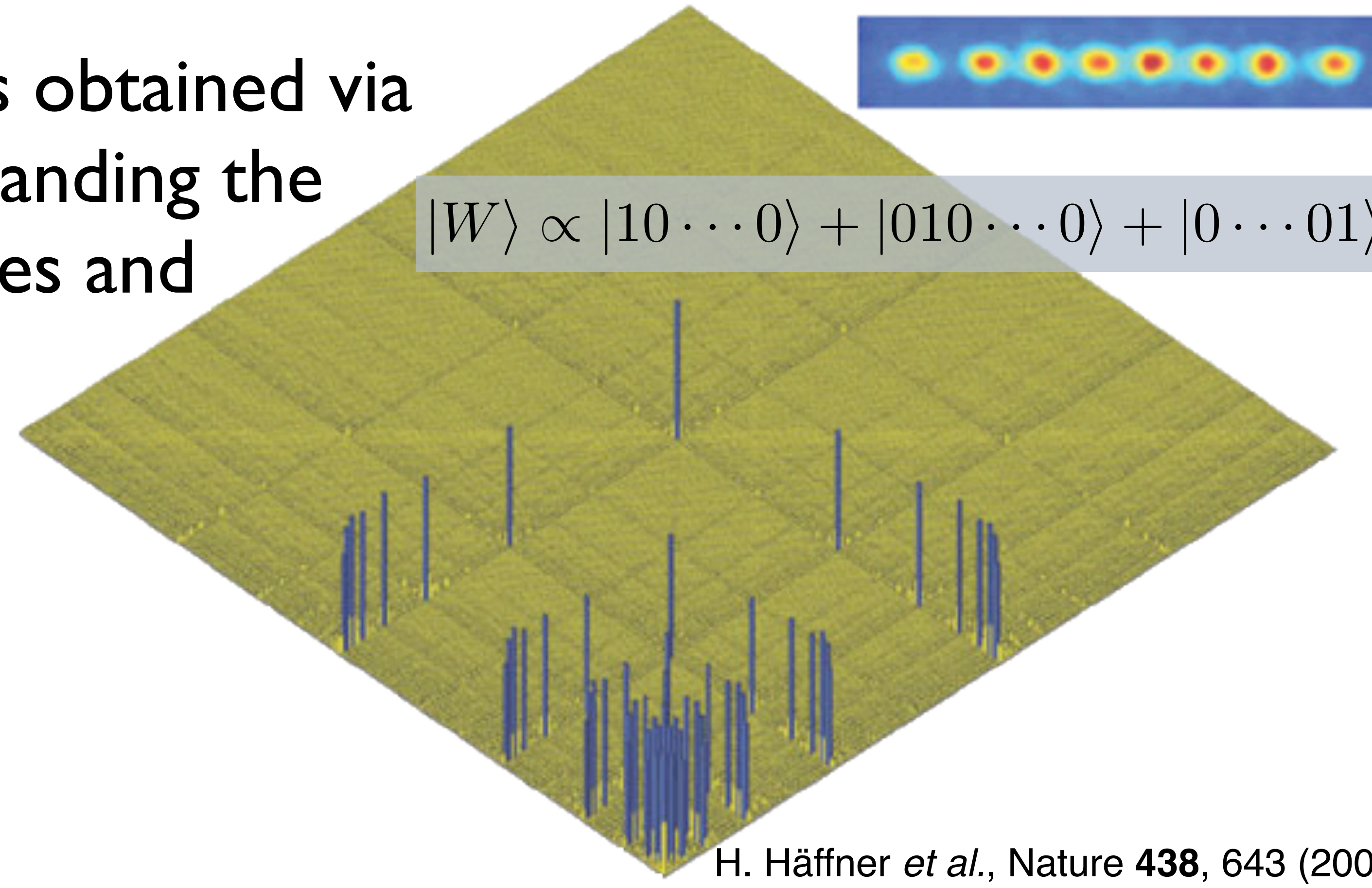

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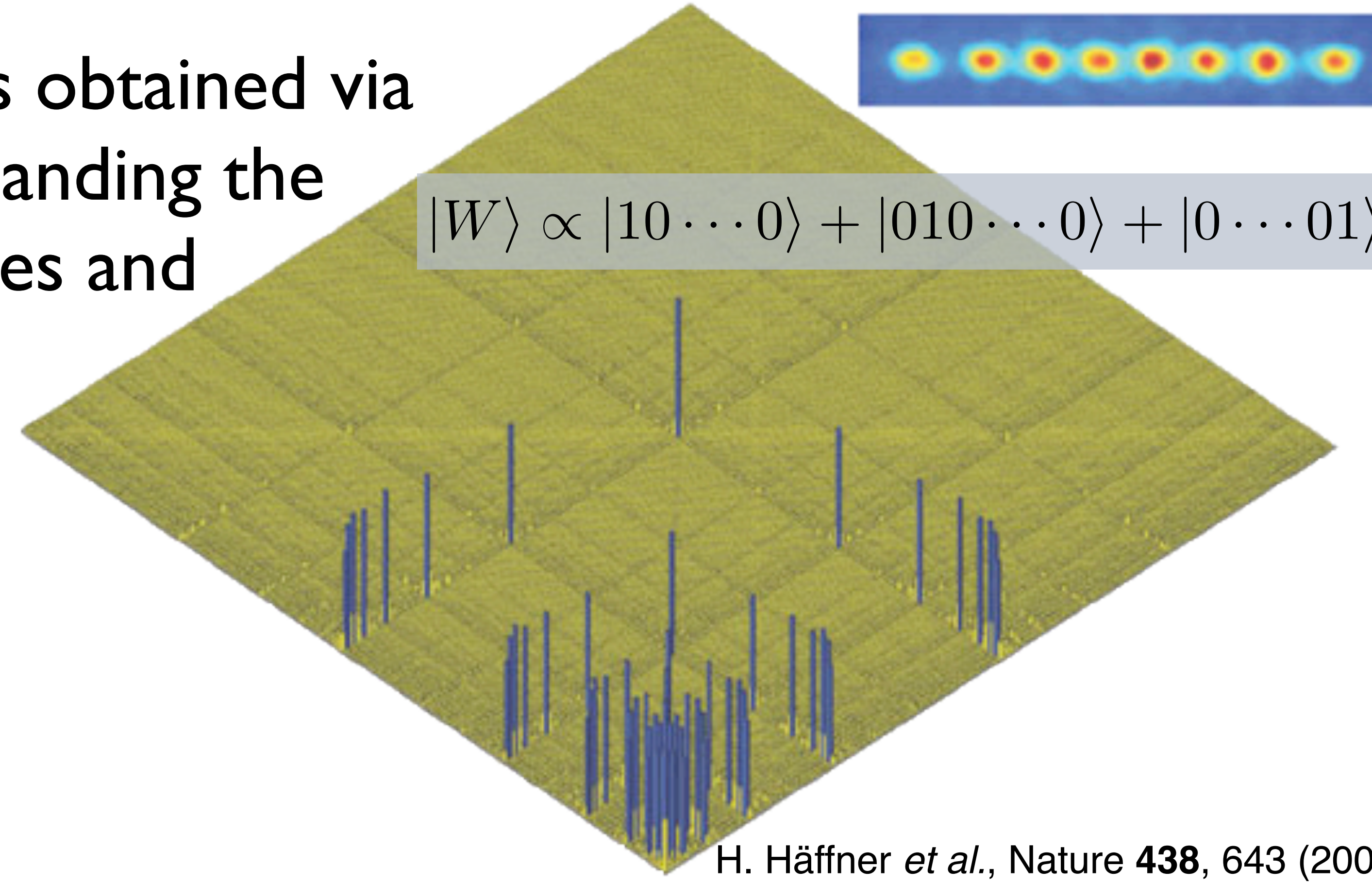

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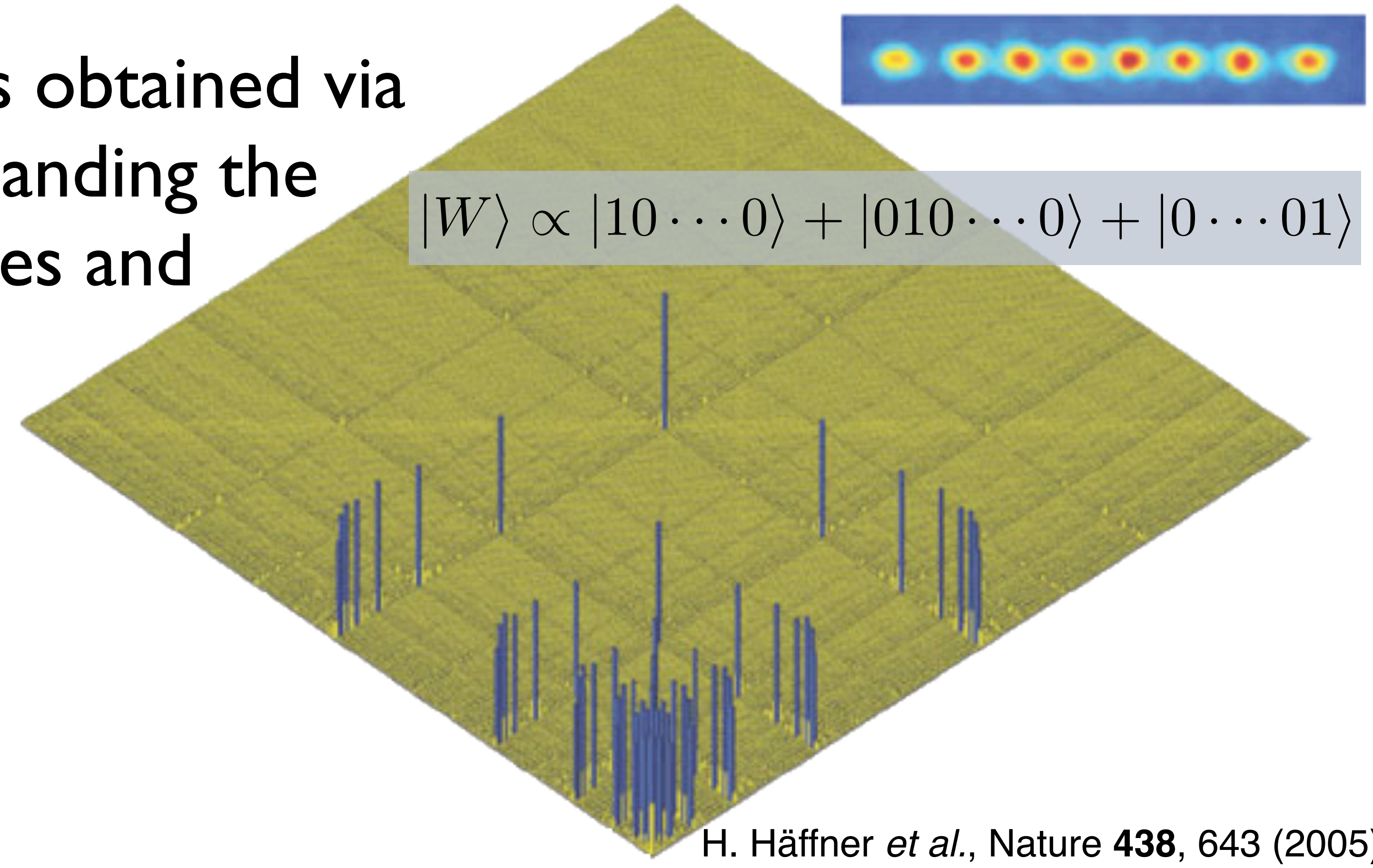

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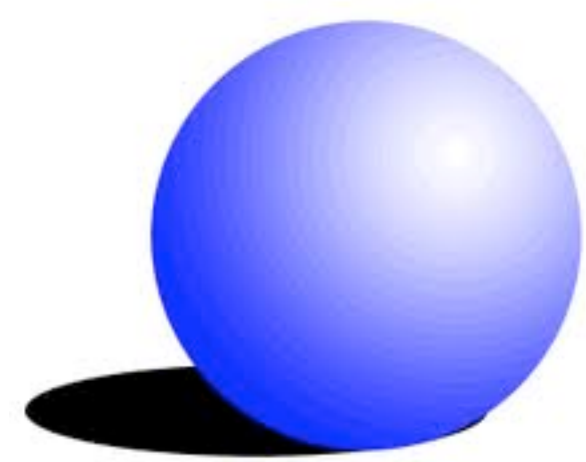

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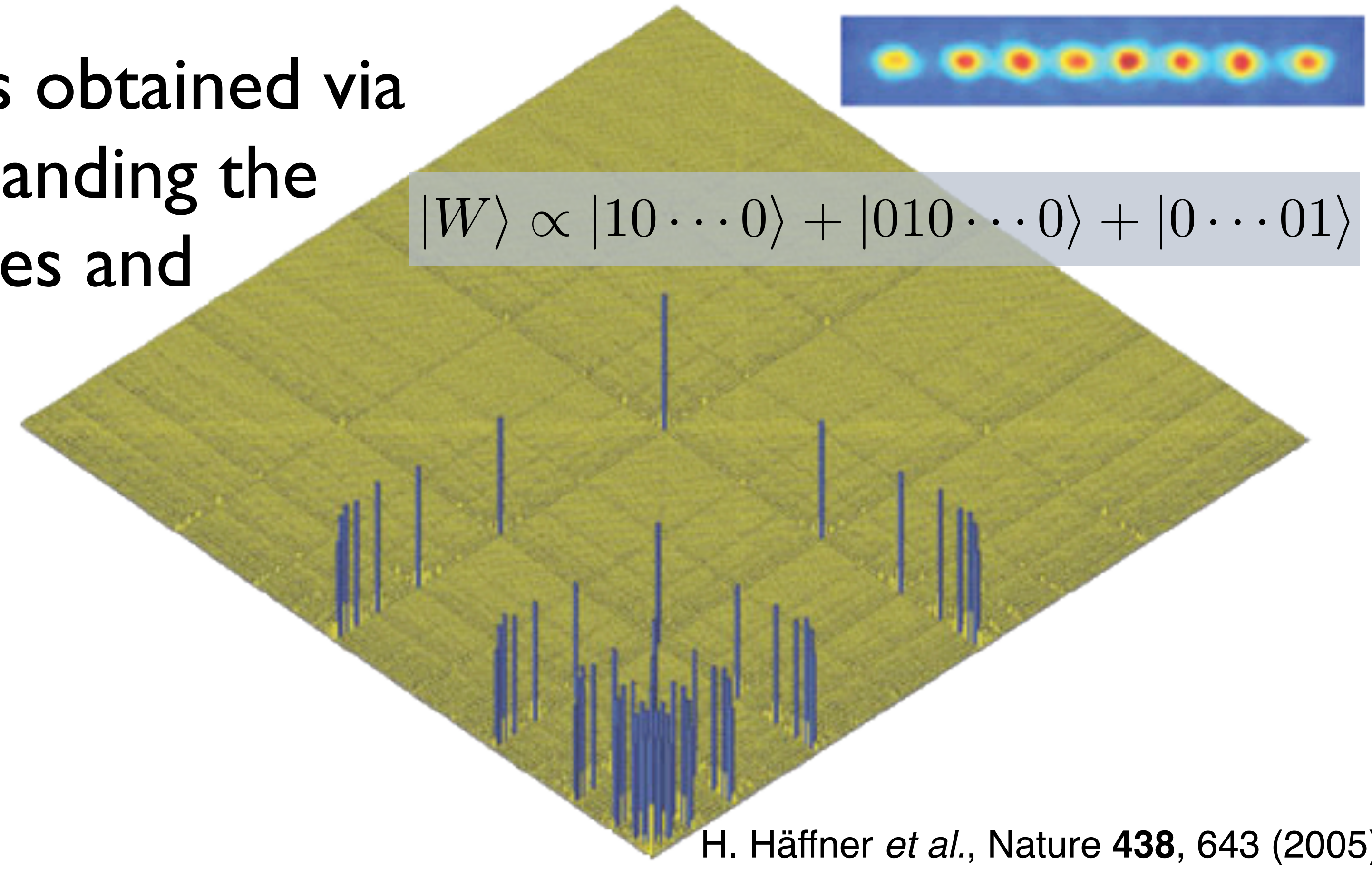

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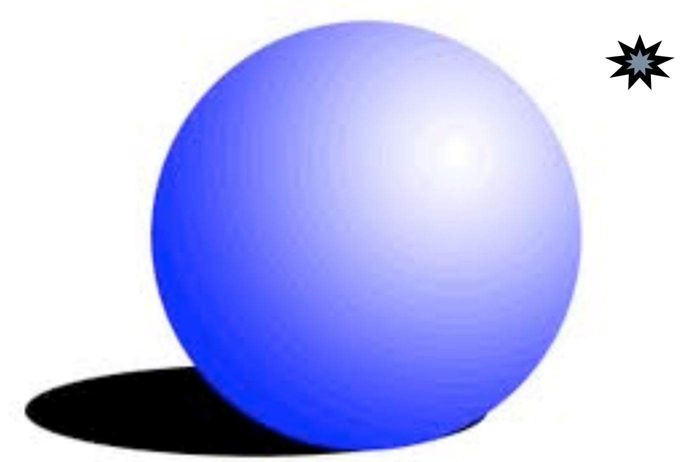

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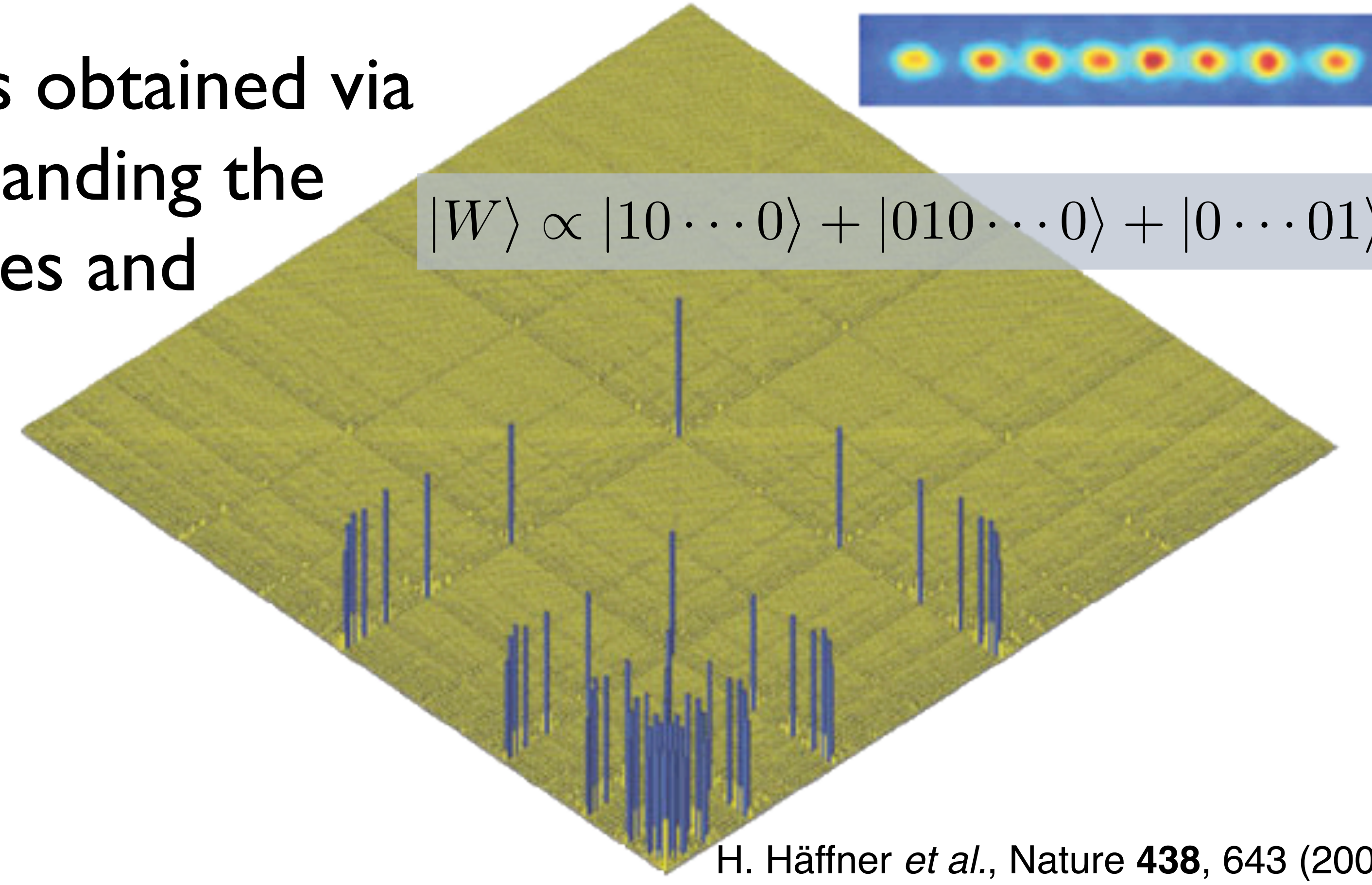

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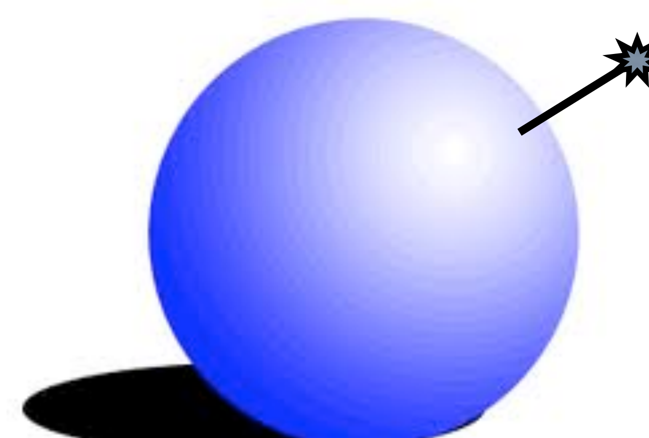

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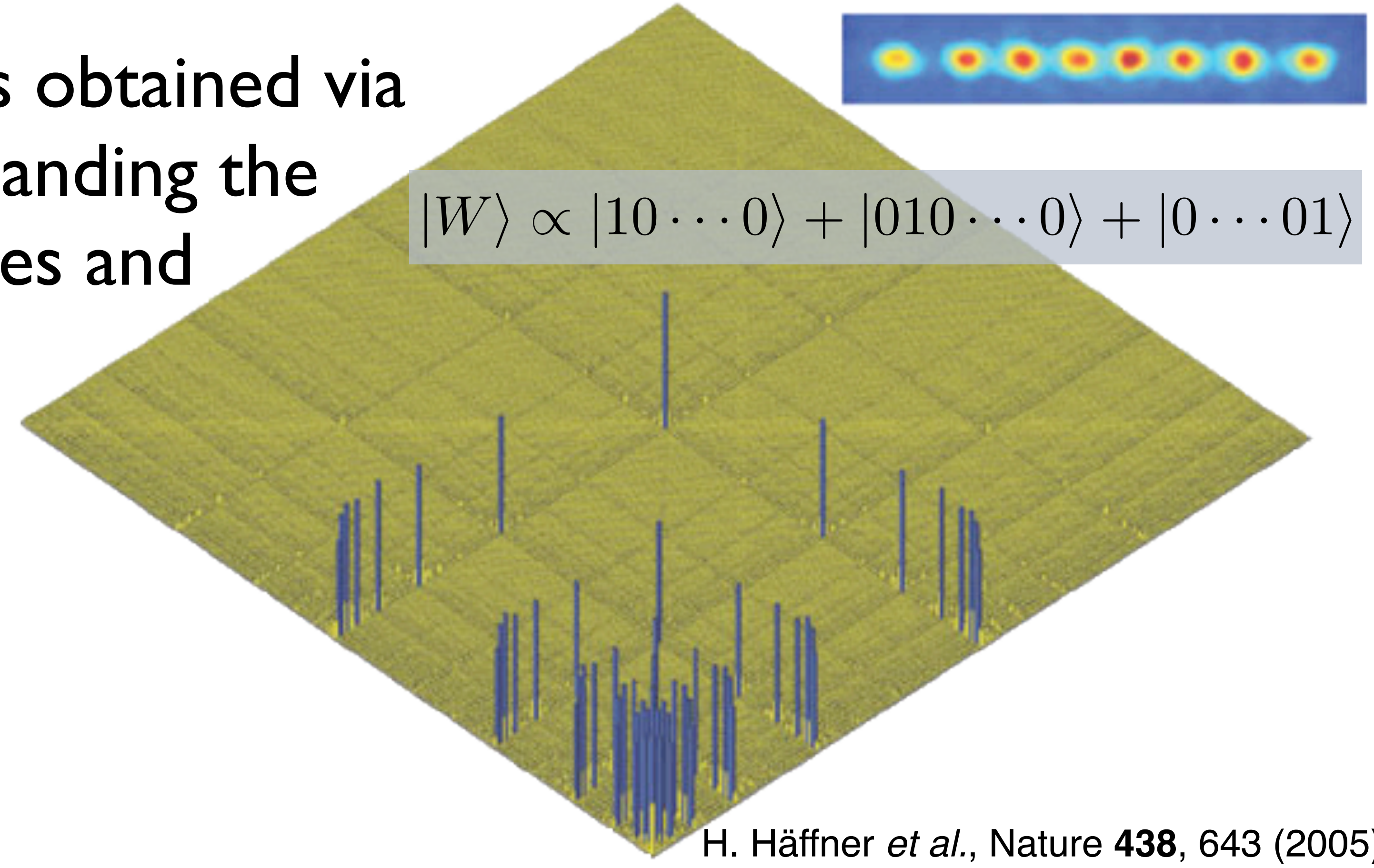

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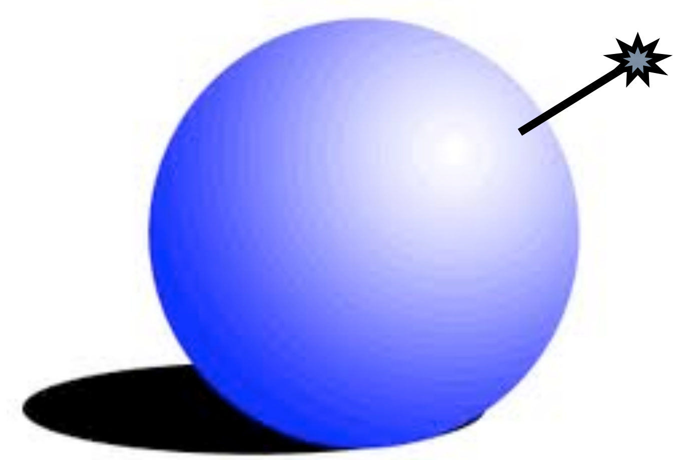

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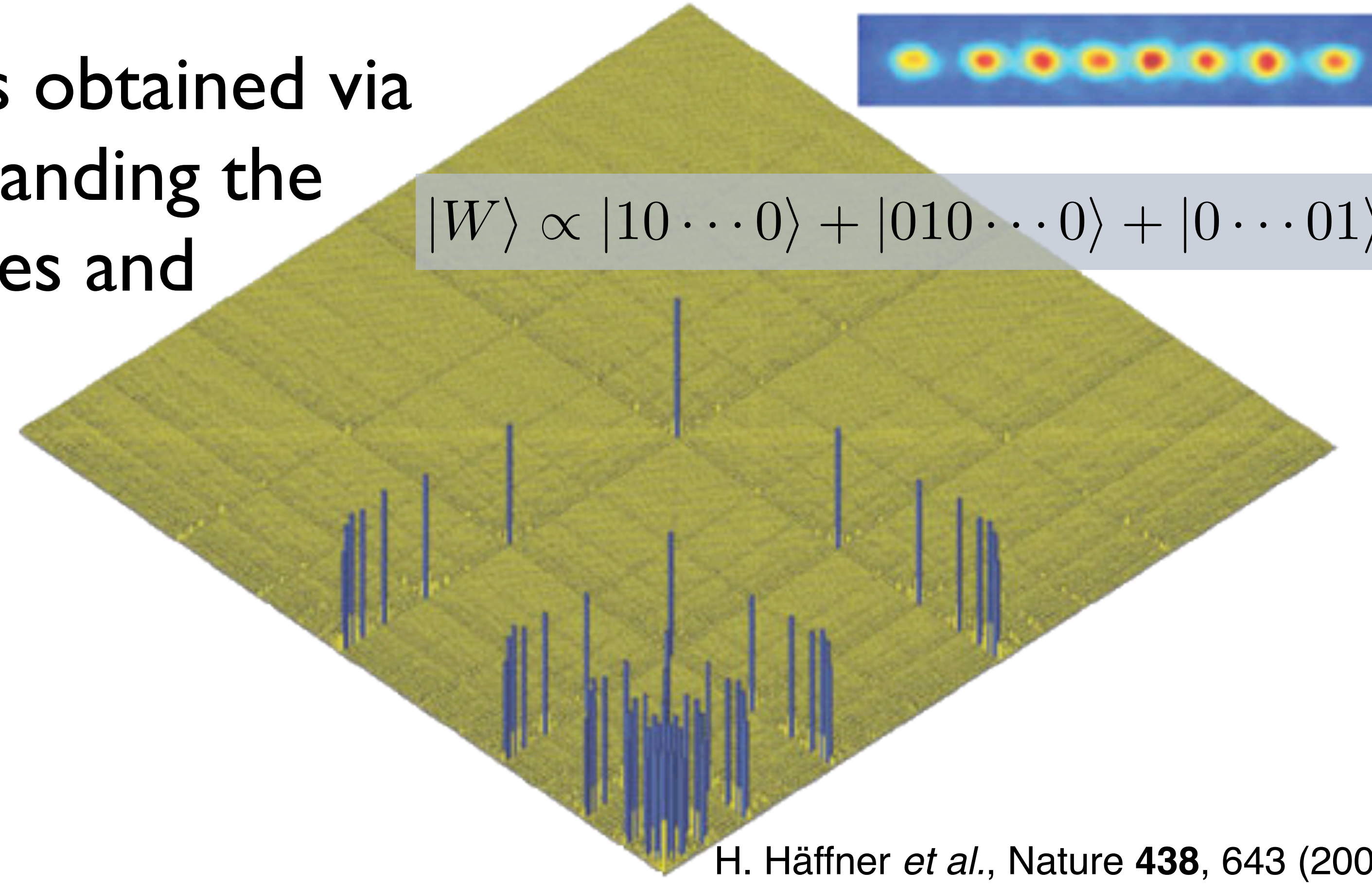

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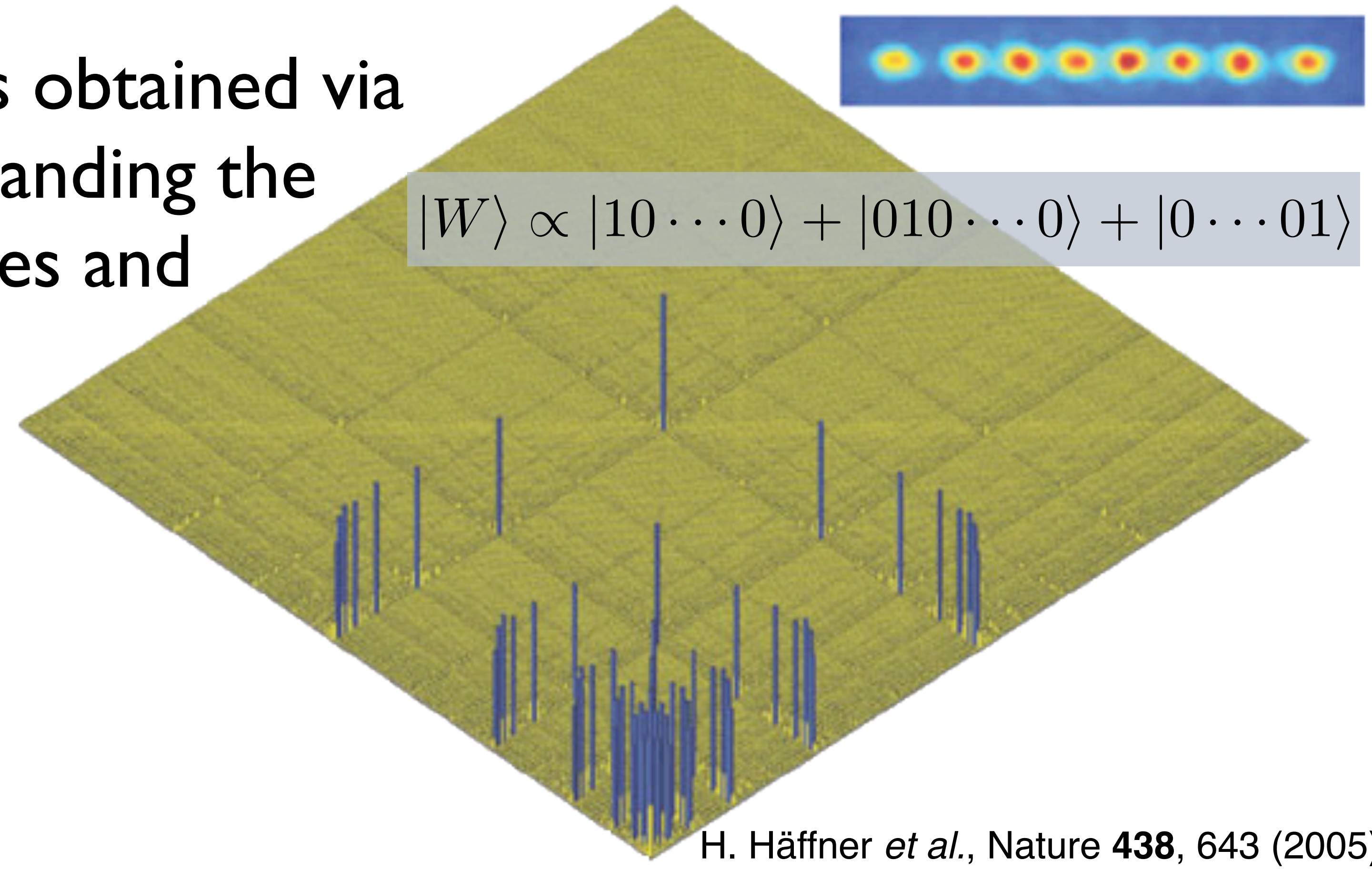

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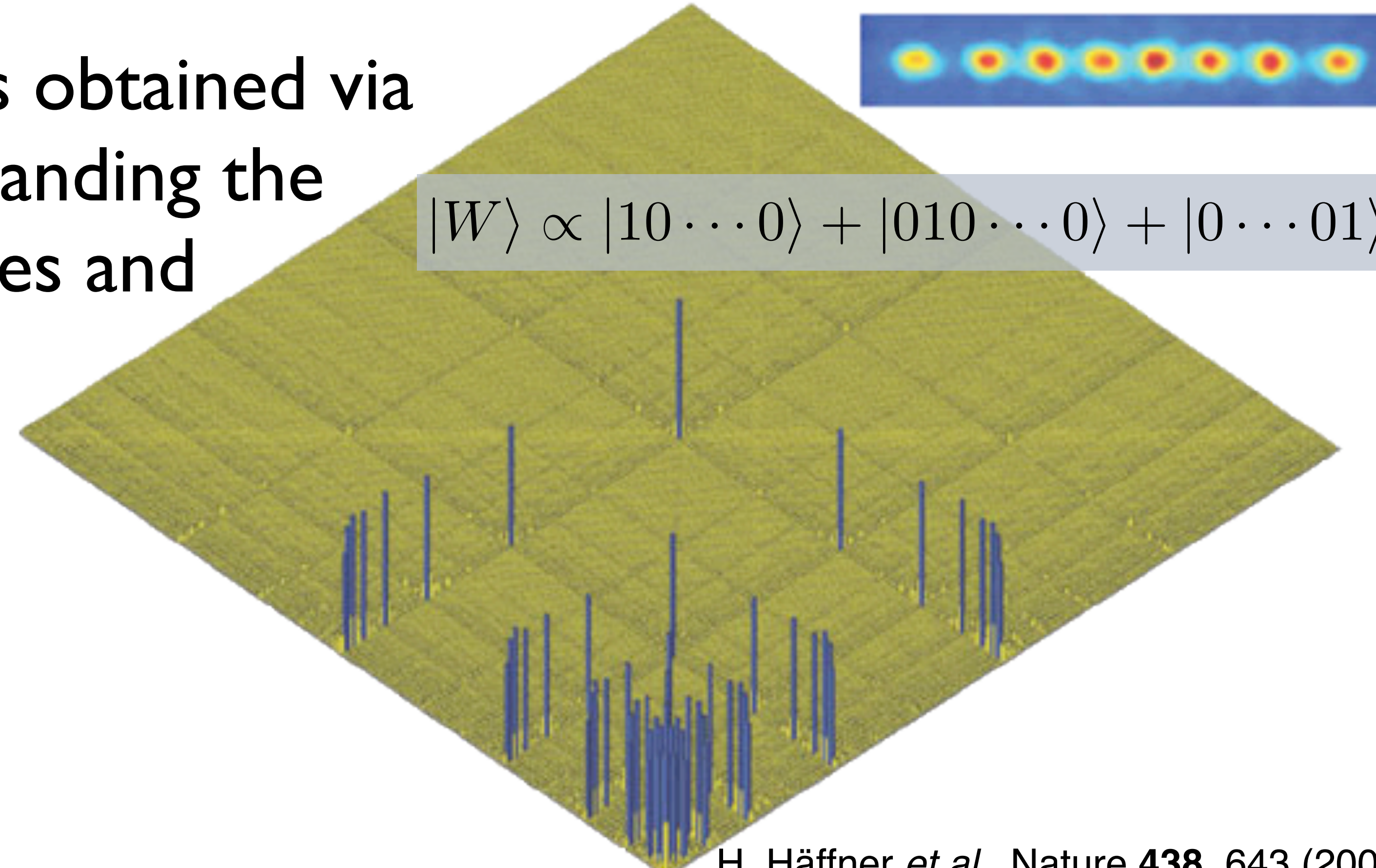

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$$\hat\sigma_1^{\alpha_1} \hat\sigma_2^{\alpha_2} \cdots \hat\sigma_N^{\alpha_N} =: \hat P_k \qquad \alpha_i = x, y, z, 0$$

$$\hat\varrho = \sum_{k=1}^{4^N} \mathrm{tr}[\hat P_k \hat\varrho] \frac{\hat P_k}{2^N}$$

4^N **observables**

$$\hat{\varrho} \propto \sum_k \langle \hat{P}_k \rangle \hat{P}_k$$

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4^N observables

- Possible for few-particle systems
- Infeasible for large systems

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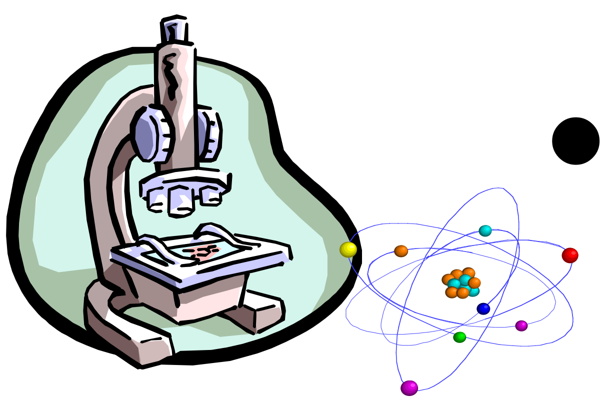

- Number and accuracy of measurements

- Find compatible state
- Storage space

State tomography: The problem

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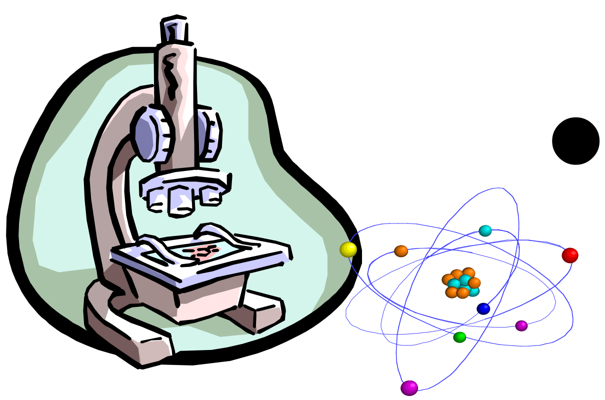

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efficient description needed

But why do full state tomography to **learn** the state?
Why not just **verify** that the intended state $|\psi\rangle$ has been created?
Why not simply measure the observable $|\psi\rangle\langle\psi|$?

$$\big\langle |\psi\rangle\langle\psi| \big\rangle = \mathrm{tr}\big[|\psi\rangle\langle\psi|\hat{\varrho}\big] = \langle\psi|\hat{\varrho}|\psi\rangle$$

$$1 - \langle\psi|\hat{\varrho}|\psi\rangle \leq \tfrac{1}{2}\big\| |\psi\rangle\langle\psi| - \hat{\varrho} \big\|_{\mathrm{tr}} \leq \sqrt{1 - \langle\psi|\hat{\varrho}|\psi\rangle}$$

$$\big| \langle\hat{A}\rangle_{\hat{\sigma}} - \langle\hat{A}\rangle_{\hat{\varrho}} \big| \leq \|\hat{A}\| \, \|\hat{\varrho} - \hat{\sigma}\|_{\mathrm{tr}}$$

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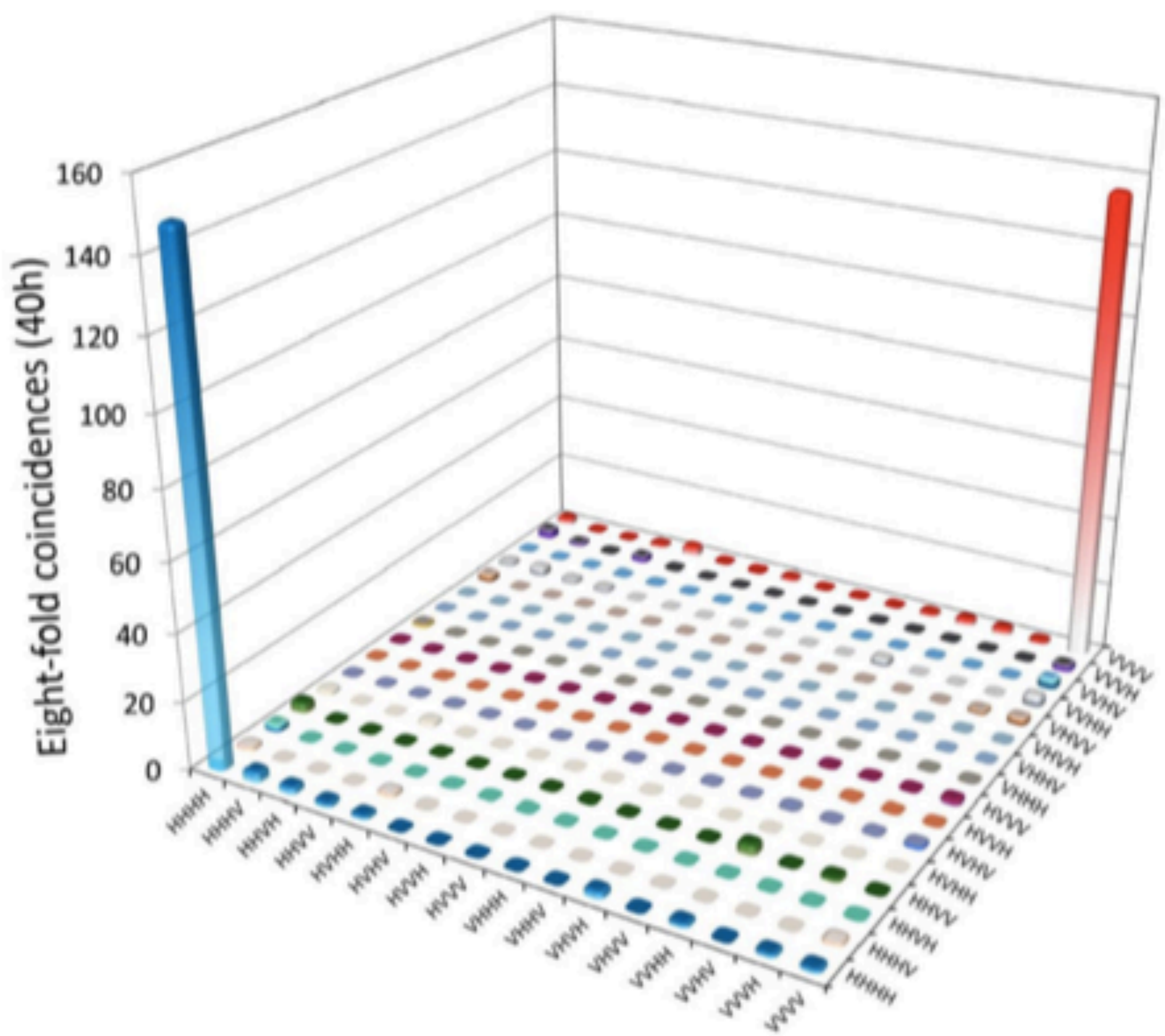
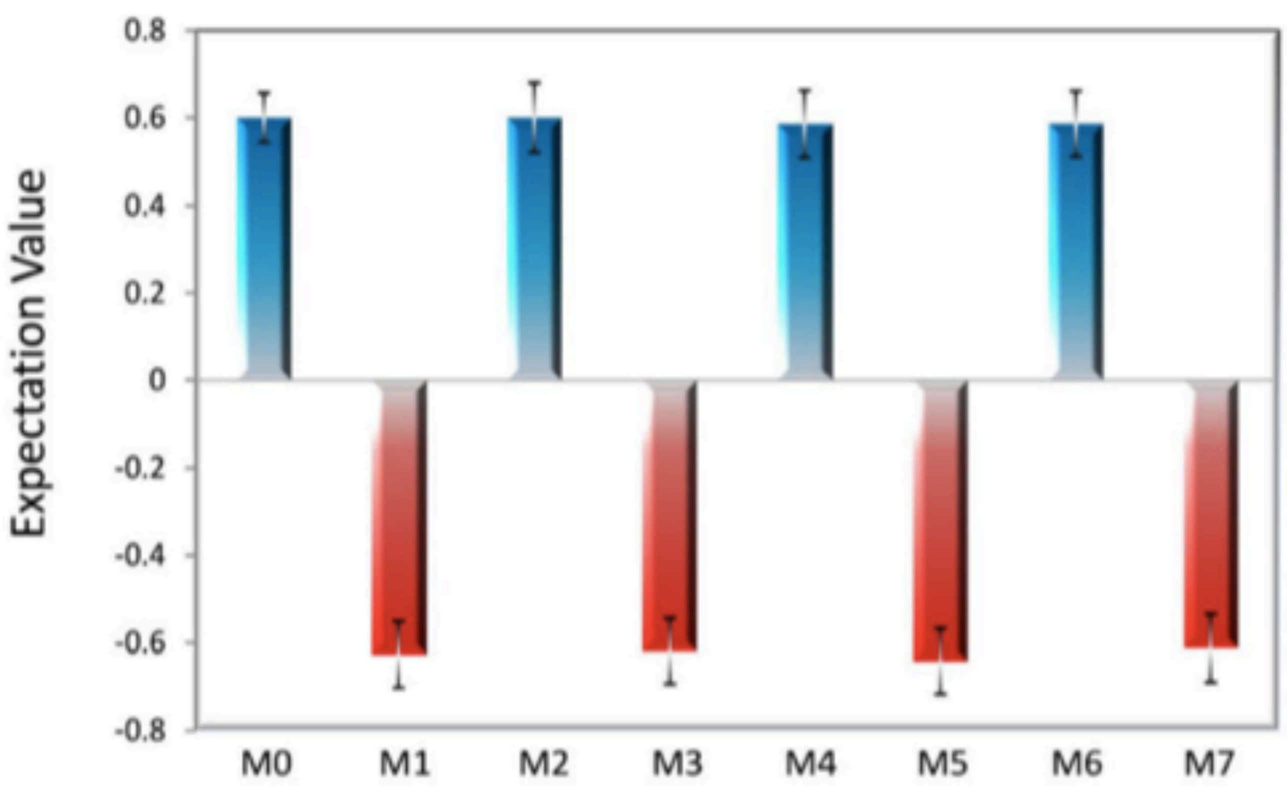
$$|GHZ\rangle\langle GHZ| = \frac{|0\cdots0\rangle\langle0\cdots0|+|1\cdots1\rangle\langle1\cdots1|}{2} + \frac{1}{16}\sum_{n=0}^{7}(-1)^n \hat{M}_n^{\otimes 8}$$

$$|GHZ\rangle \propto |0\cdots0\rangle + |1\cdots1\rangle$$





Yao *et al.*, arXiv:1105.6318

Where are we at? - N=9

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$$\hat{P}_k = \bigotimes_{i=1}^{N} \hat{\sigma}_i^{\alpha_i}$$

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ARTICLE

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Onset of a quantum phase transition with a trapped ion quantum simulator

R. Islam[1], E.E. Edwards[1], K. Kim[1], S. Korenblit[1], C. Noh[2], H. Carmichael[2], G.-D. Lin[3], L.-M. Duan[3], C.-C. Joseph Wang[4], J.K. Freericks[4] & C. Monroe[1]

A quantum simulator is a well-controlled quantum system that can follow the evolution of a prescribed model whose behaviour may be difficult to determine. A good example is the simulation of a set of interacting spins, where phase transitions between various spin orders can underlie poorly understood concepts such as spin liquids. Here we simulate the emergence of magnetism by implementing a fully connected non-uniform ferromagnetic quantum Ising model using up to 9 trapped $^{171}Yb^{+}$ ions. By increasing the Ising coupling strengths compared with the transverse field, the crossover from paramagnetism to ferromagnetic order sharpens as the system is scaled up, prefacing the expected quantum phase transition in the

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PRL **106,** 130506 (2011) PHYSICAL REVIEW LETTERS week ending 1 APRIL 2011

14-Qubit Entanglement: Creation and Coherence

Thomas Monz,[1] Philipp Schindler,[1] Julio T. Barreiro,[1] Michael Chwalla,[1] Daniel Nigg,[1] William A. Coish,[2,3] Maximilian Harlander,[1] Wolfgang Hänsel,[4] Markus Hennrich,[1,*] and Rainer Blatt[1,4]
[1]*Institut für Experimentalphysik, Universität Innsbruck, Technikerstr. 25, A-6020 Innsbruck, Austria*
[2]*Institute for Quantum Computing and Department of Physics and Astronomy, University of Waterloo, Waterloo, ON, N2L 3G1, Canada*
[3]*Department of Physics, McGill University, Montreal, Quebec, Canada H3A 2T8*
[4]*Institut für Quantenoptik und Quanteninformation, Österreichische Akademie der Wissenschaften, Otto-Hittmair-Platz 1, A-6020 Innsbruck, Austria*
(Received 30 September 2010; published 31 March 2011)

We report the creation of Greenberger-Horne-Zeilinger states with up to 14 qubits. By investigating the coherence of up to 8 ions over time, we observe a decay proportional to the square of the number of qubits. The observed decay agrees with a theoretical model which assumes a system affected by correlated, Gaussian phase noise. This model holds for the majority of current experimental systems developed towards quantum computation and quantum metrology.

TABLE I. Populations, coherence, and fidelity with a N-qubit GHZ state of experimentally prepared states. Entanglement criteria supported by σ standard deviations. All errors in parenthesis, 1 standard deviation.

Number of ions	2	3	4	5	6	8	10	12	14
Populations, %	99.50(7)	97.6(2)	97.5(2)	96.0(4)	91.6(4)	84.7(4)	67.0(8)	53.3(9)	56.2(11)
Coherence, %	97.8(3)	96.5(6)	93.9(5)	92.9(8)	86.8(8)	78.7(7)	58.2(9)	41.6(10)	45.4(13)
Fidelity, %	**98.6(2)**	**97.0(3)**	**95.7(3)**	**94.4(5)**	**89.2(4)**	**81.7(4)**	**62.6(6)**	**47.4(7)**	**50.8(9)**
Distillability criterion [14], σ	283	151	181	100	95	96	40	18	17

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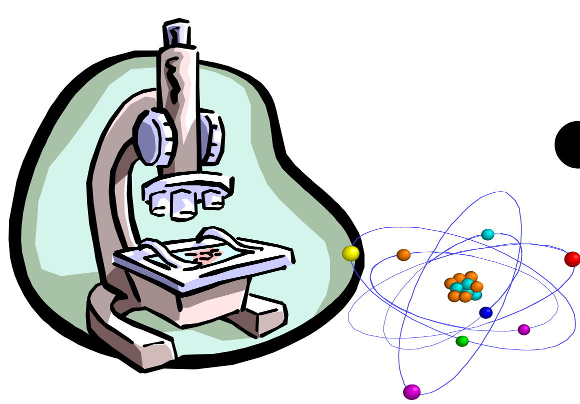

- Number and accuracy of measurements

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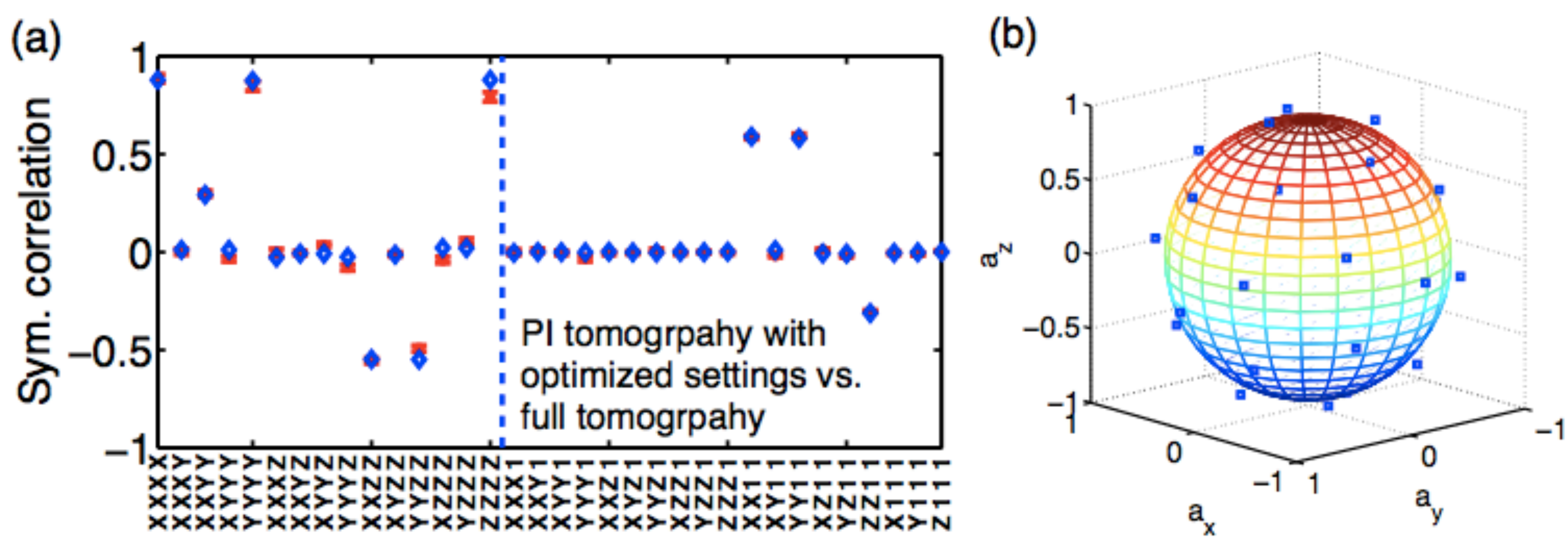
efficient description needed

Permutationally invariant tomography

$$\hat{\varrho}_{\rm PI} = \frac{1}{N!}\sum_k \hat{\pi}_k \hat{\varrho} \hat{\pi}_k$$

$$|D\rangle \propto |1100\rangle + |1010\rangle + |1001\rangle + |0110\rangle + |0101\rangle + |0011\rangle$$



Toth, Wieczorek, Gross, Krischek, Schwemmer, Weinfurter, PRL **105**, 250403 (2010)

$$\hat{\varrho}_{\mathrm{PI}} = \frac{1}{N!}\sum_k \hat{\pi}_k \hat{\varrho} \hat{\pi}_k$$

- Basis: $\left(\hat{\sigma}_x^{\otimes k} \otimes \hat{\sigma}_y^{\otimes l} \otimes \hat{\sigma}_z^{\otimes m} \otimes \mathbb{1}^{\otimes n}\right)_{\mathrm{PI}}$

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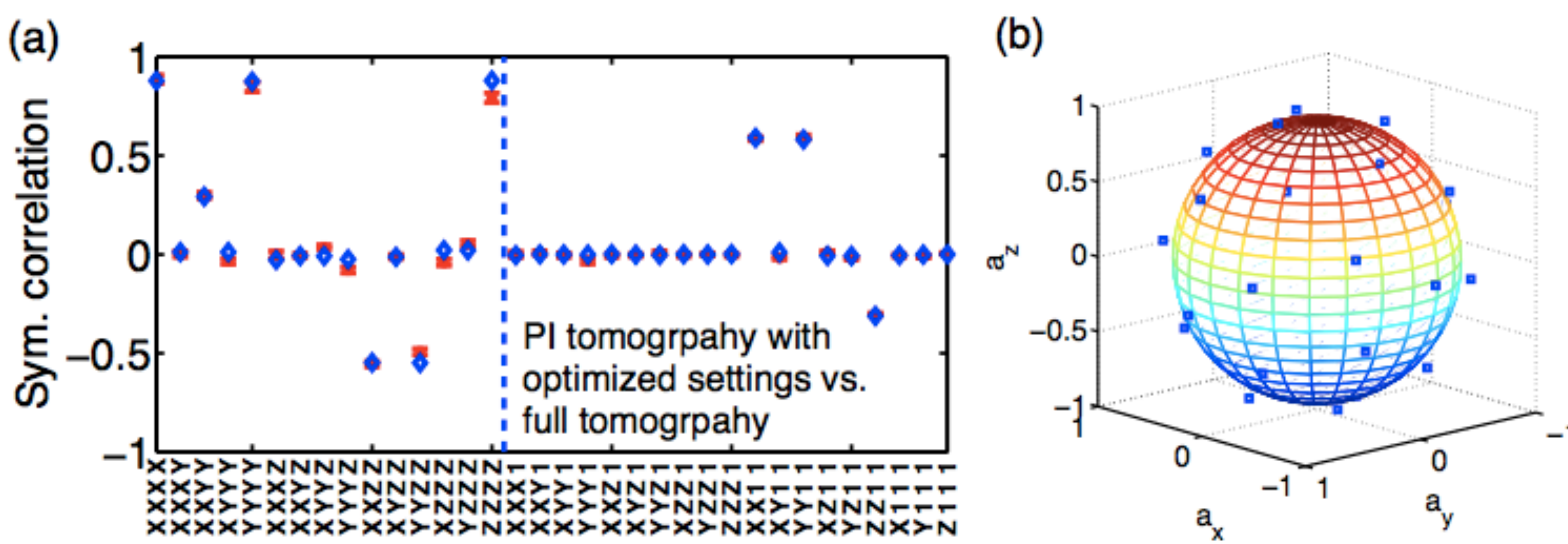

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- Characterized by $\sim N^3$ observables, “entries”

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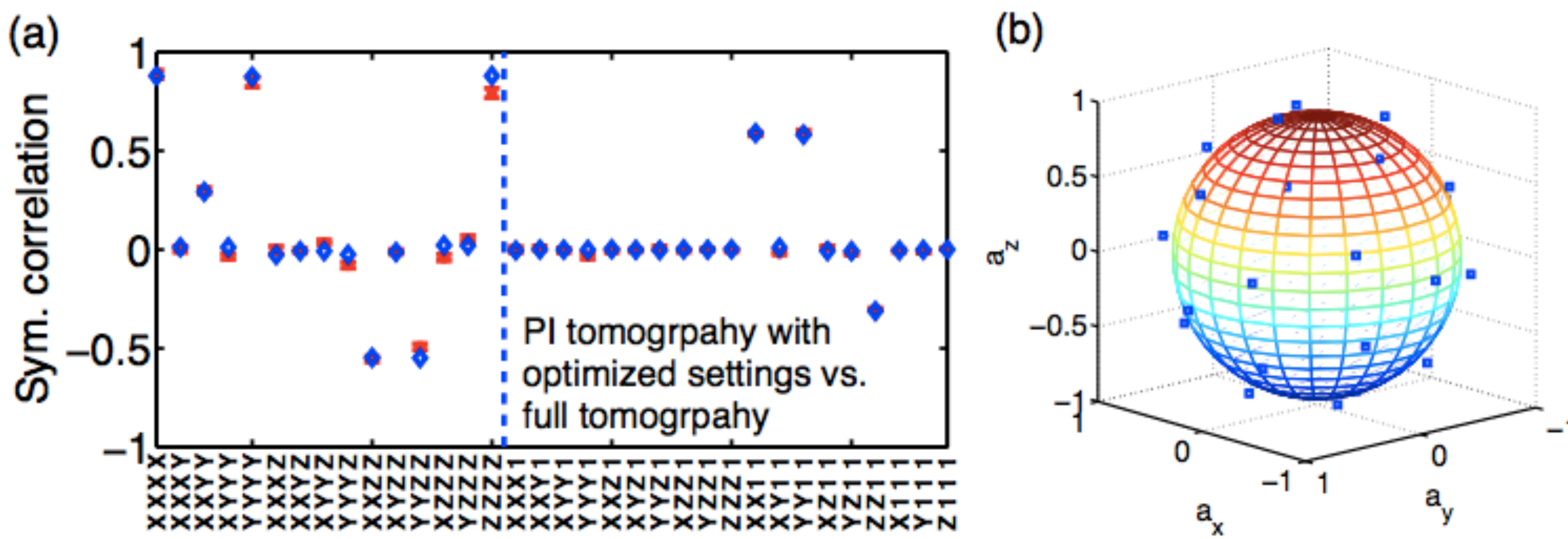


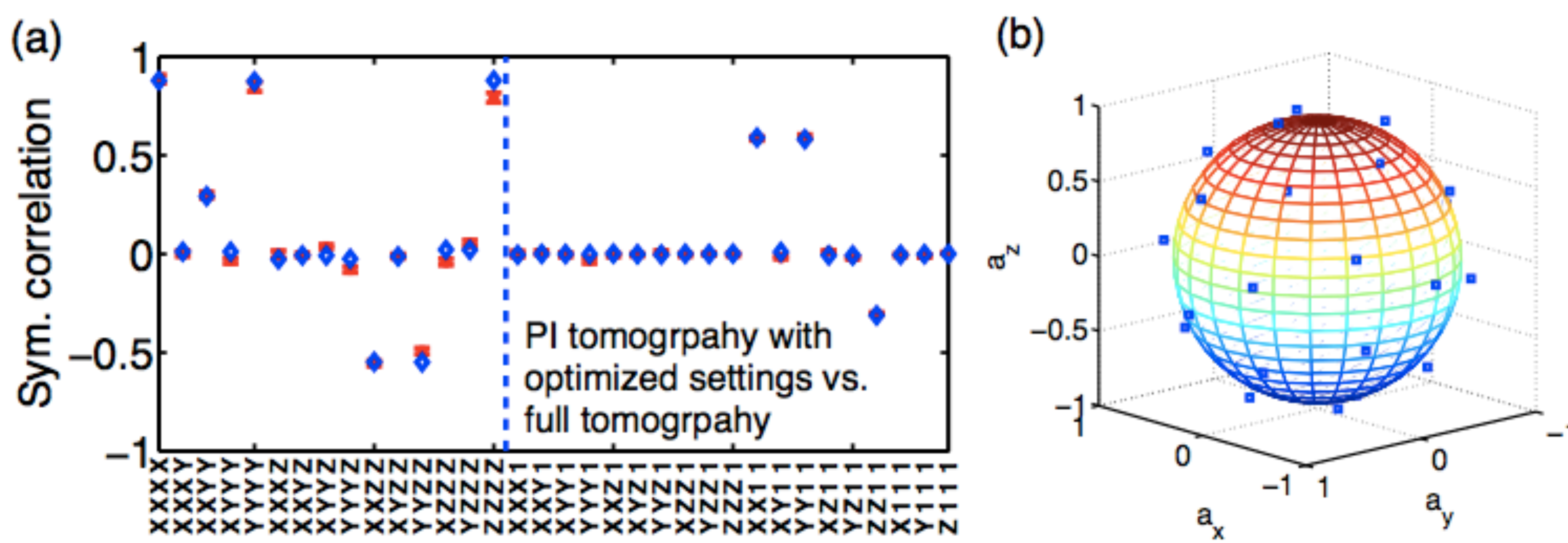
Toth, Wieczorek, Gross, Krischek, Schwemmer, Weinfurter, PRL **105**, 250403 (2010)

Permutationally invariant tomography

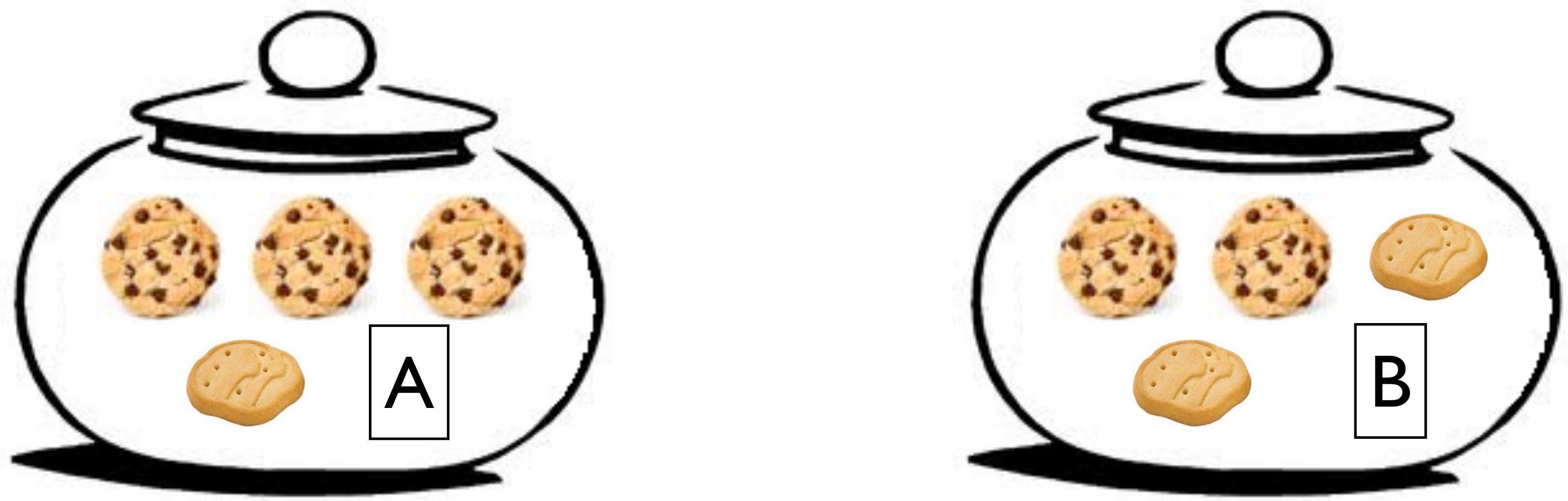
$$\hat{\varrho}_{\mathrm{PI}} = \frac{1}{N!} \sum_k \hat{\pi}_k \hat{\varrho} \hat{\pi}_k$$

- Basis: $\left(\hat{\sigma}_x^{\otimes k} \otimes \hat{\sigma}_y^{\otimes l} \otimes \hat{\sigma}_z^{\otimes m} \otimes \mathbb{1}^{\otimes n}\right)_{\mathrm{PI}}$
- Characterized by $\sim N^3$ observables, "entries"
- Certificate given

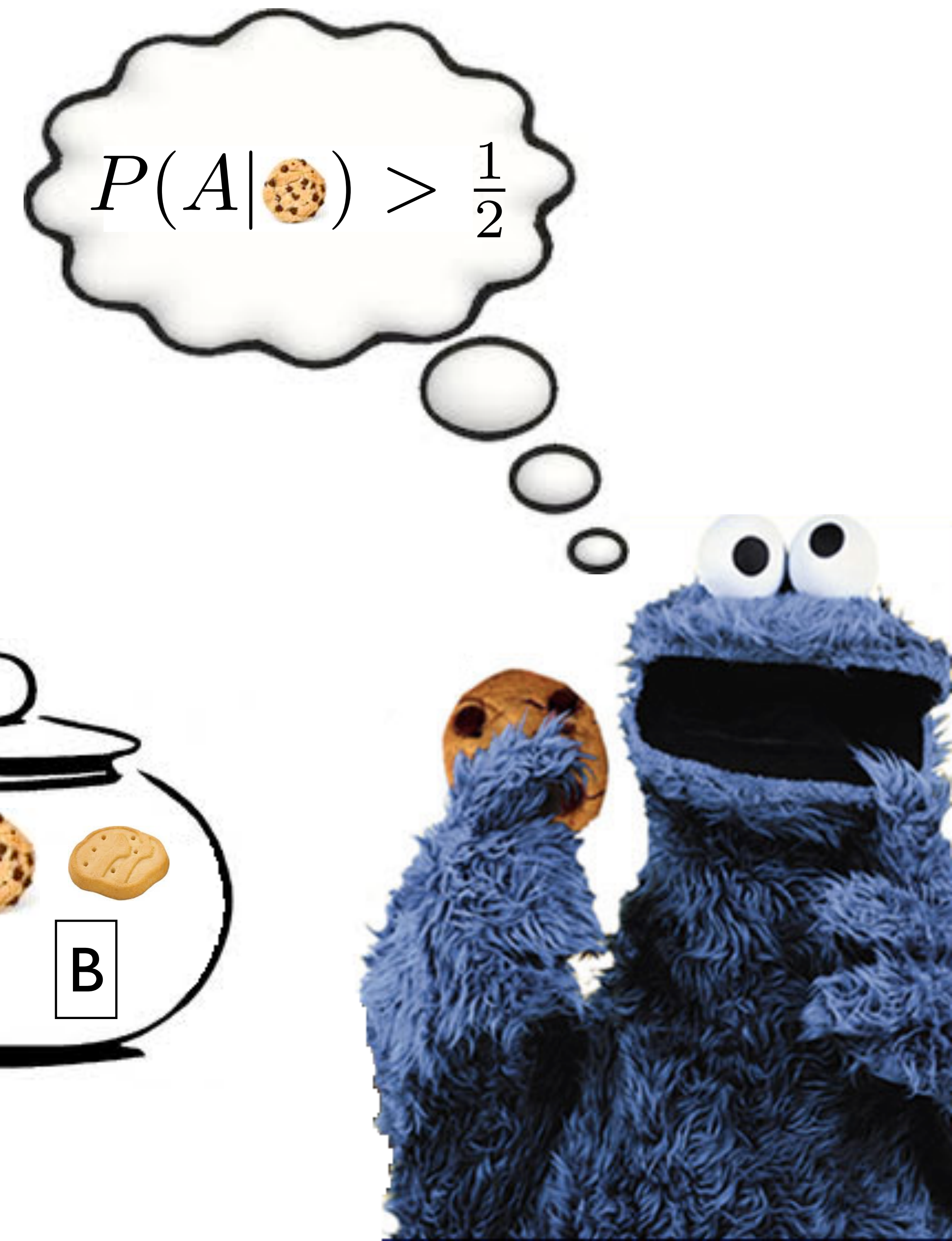
$$|D\rangle \propto |1100\rangle + |1010\rangle + |1001\rangle + |0110\rangle + |0101\rangle + |0011\rangle$$

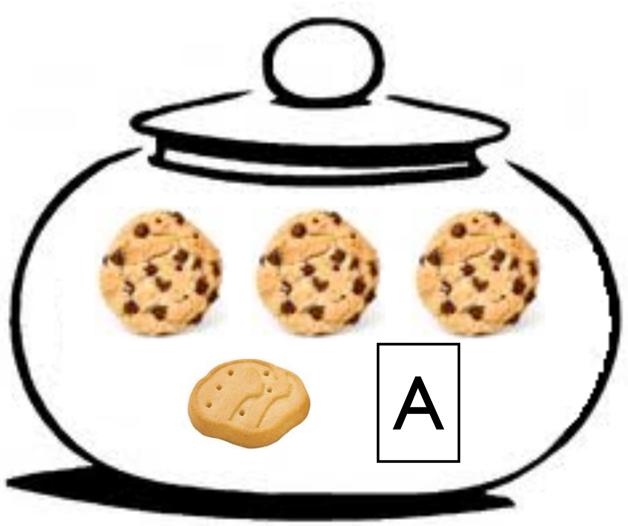


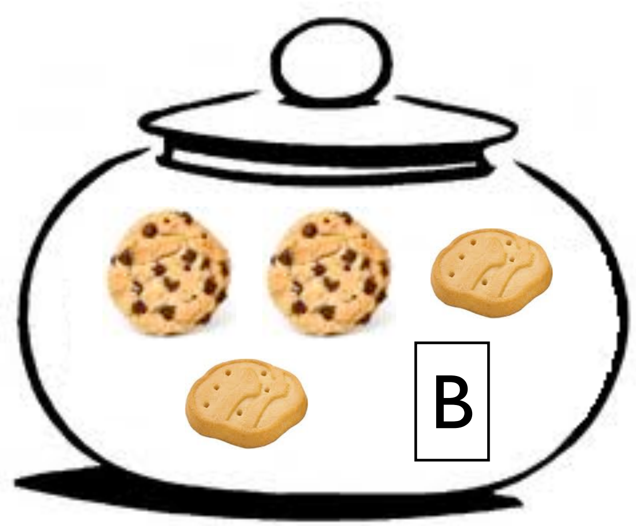
Toth, Wieczorek, Gross, Krischek, Schwemmer, Weinfurter, PRL **105**, 250403 (2010)



$$P(A) = P(B) = \frac{1}{2}$$







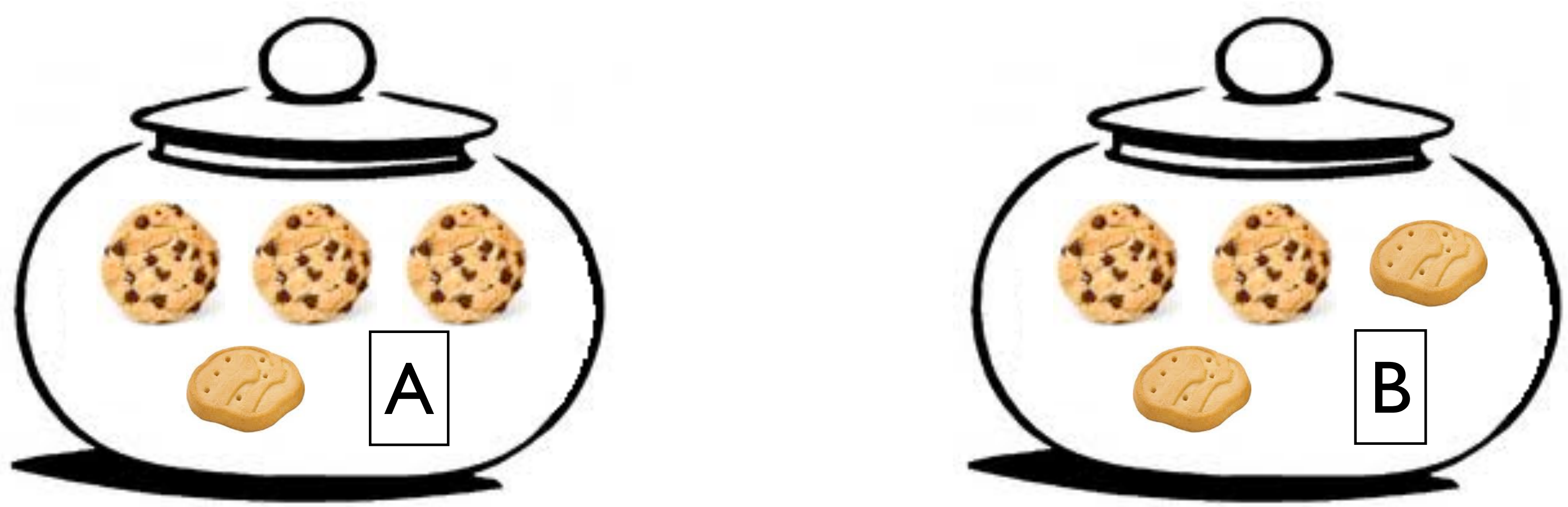
$$P(A) = P(B) = \frac{1}{2}$$

Bayes' theorem: $P(\theta|E) = \frac{P(\theta)P(E|\theta)}{P(E)}$

posterior

prior

likelihood

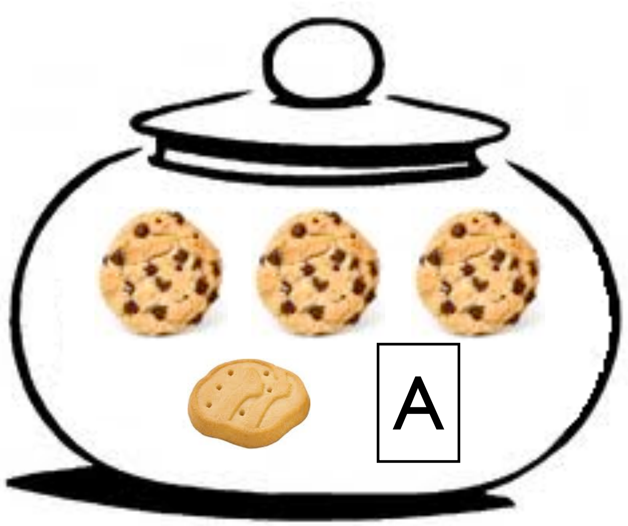


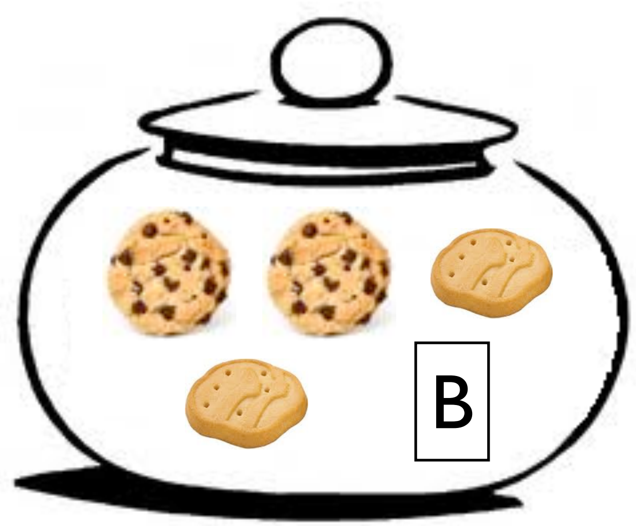
$$P(A) = P(B) = \frac{1}{2}$$

Bayes' theorem: $P(\theta|E) = \dfrac{P(\theta)P(E|\theta)}{P(E)}$

$$P(A|🍪) = \frac{P(A)P(🍪|A)}{P(🍪)} = \frac{P(A)P(🍪|A)}{P(A)P(🍪|A) + P(B)P(🍪|B)}$$

$$= \frac{\frac{1}{2}\cdot\frac{3}{4}}{\frac{1}{2}\cdot\frac{3}{4} + \frac{1}{2}\cdot\frac{2}{4}} = 0.6$$





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Sample N times, obtain sample means g_i, $i = 1, \ldots, R$

$$P(\hat{\varrho}|\{g_i\}) \propto P(\{g_i\}|\hat{\varrho})P(\hat{\varrho})$$

J. Rau, PRA **82**, 012104 (2010)

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J. Rau, PRA **82**, 012104 (2010)

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$$\hat{\varrho}_e \propto \exp\left[\tfrac{\alpha}{\alpha+N}\log(\hat{\sigma}) + (1 - \tfrac{\alpha}{\alpha+N})\log(\hat{\mu})\right]$$

$$\hat{\mu} = \exp\left[\log(\hat{\sigma}) + \sum_i \lambda_i \hat{G}_i\right]$$ **s.t.** $\mathrm{tr}[\hat{\mu}\hat{G}_i] = g_i$; $\frac{\alpha}{\alpha+N} = \frac{R}{2NS(\hat{\mu}\|\hat{\sigma})}$

J. Rau, PRA **82**, 012104 (2010)

$$\hat{\varrho} \propto \sum_k \langle \hat{P}_k \rangle \hat{P}_k$$

4^N observables

$$\hat{P}_k = \bigotimes_{i=1}^{N} \hat{\sigma}_i^{\alpha_i}$$

Can we get away with less?

$$\hat{\varrho} \propto \sum_k \langle \hat{P}_k \rangle \hat{P}_k$$

4^N observables

$$\hat{P}_k = \bigotimes_{i=1}^{N} \hat{\sigma}_i^{\alpha_i}$$

Can we get away with less?

→ Construct matrix from "few" known entries

→ *Compressed sensing*

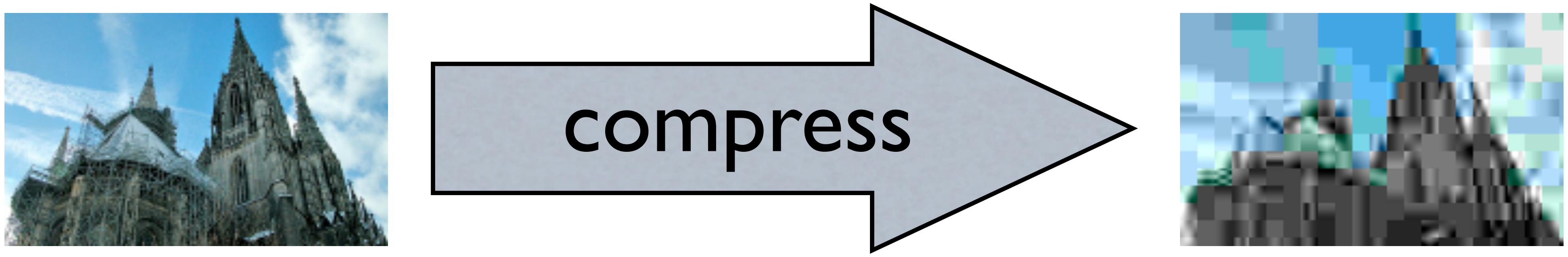
"a technique that may be the hottest topic in applied math today"

"paradigm-busting field in mathematics that's reshaping the way people work with large data sets"

"Only six years old, compressed sensing has already inspired more than a thousand papers and pulled in millions of dollars in federal grants"

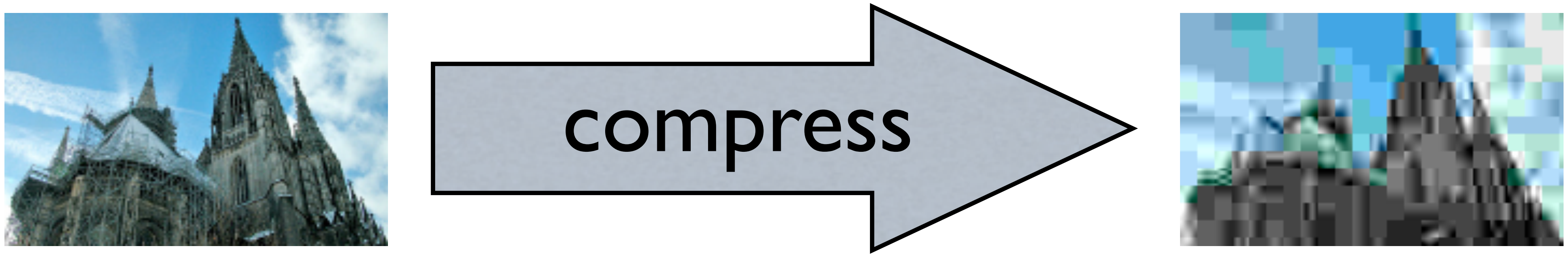
WIRED February 2010

Compressed sensing



Pioneered by Donoho, Candès, Tao;
an introduction: Candes, Wakin, IEEE Sig. Proc. Mag. **25**, 21 (2008);
a quantum version: Gross, Liu, Flammia, Becker, Eisert, PRL **105**, 150401 (2010)

Compressed sensing



instead:

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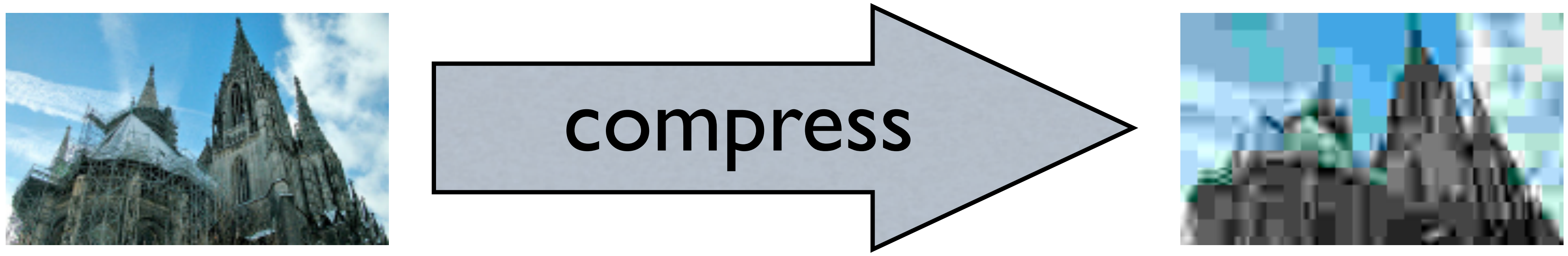
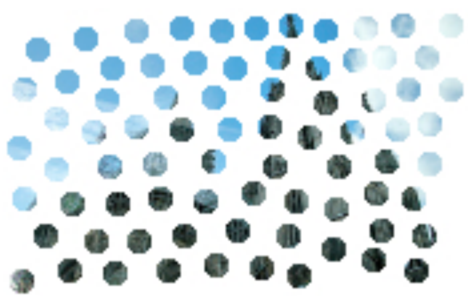
Compressed sensing



instead:



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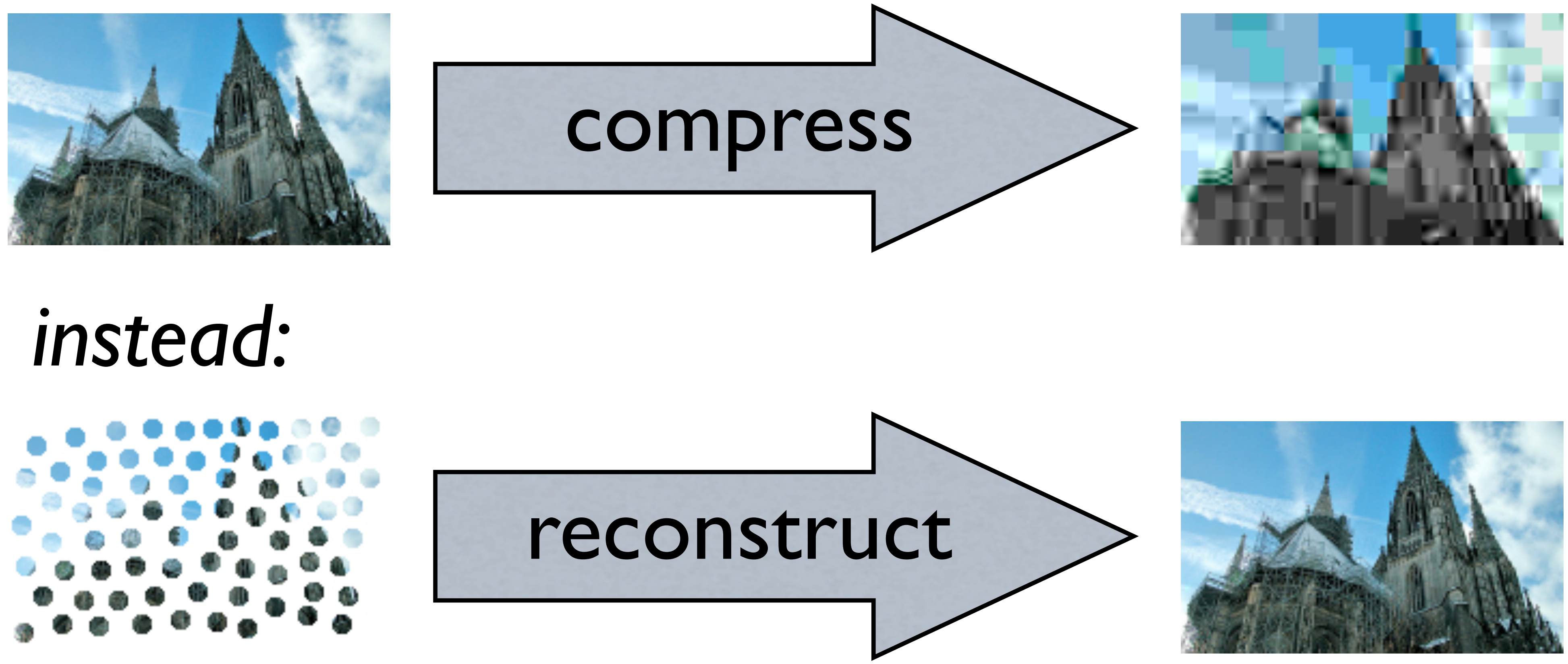
Compressed sensing



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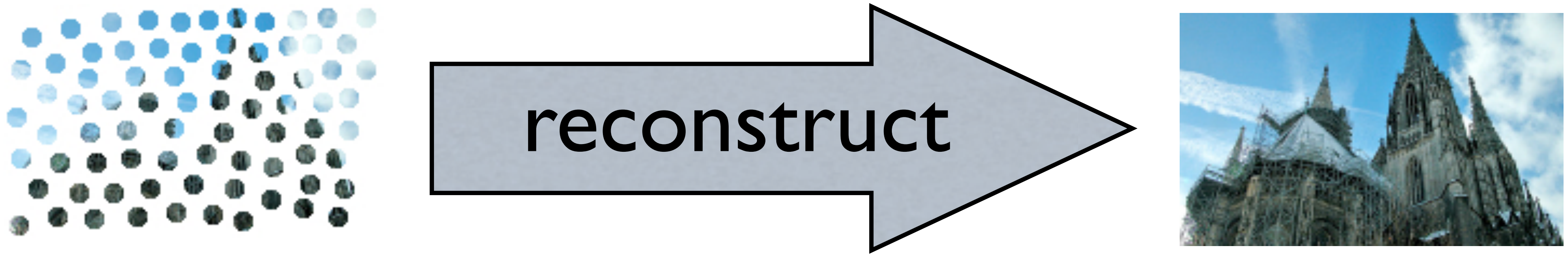
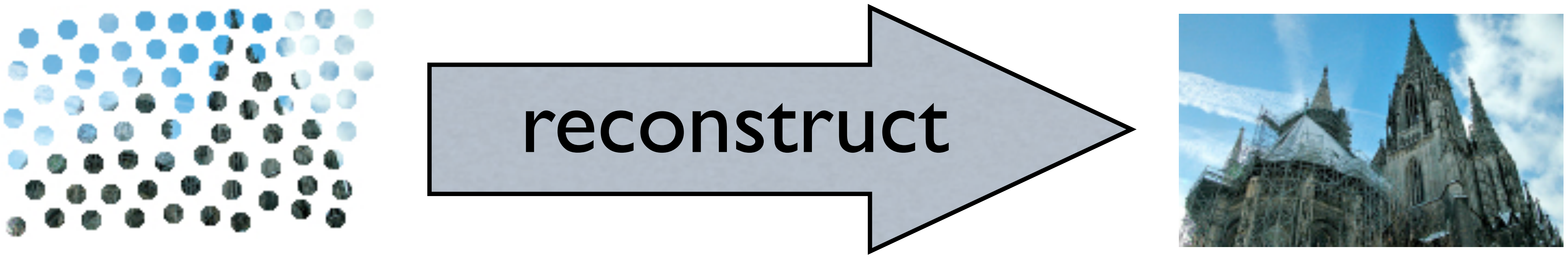


$= M$

$m \times m$

$\mathrm{rank}[M] = r$ specified by $\sim rm$ parameters



$= M$

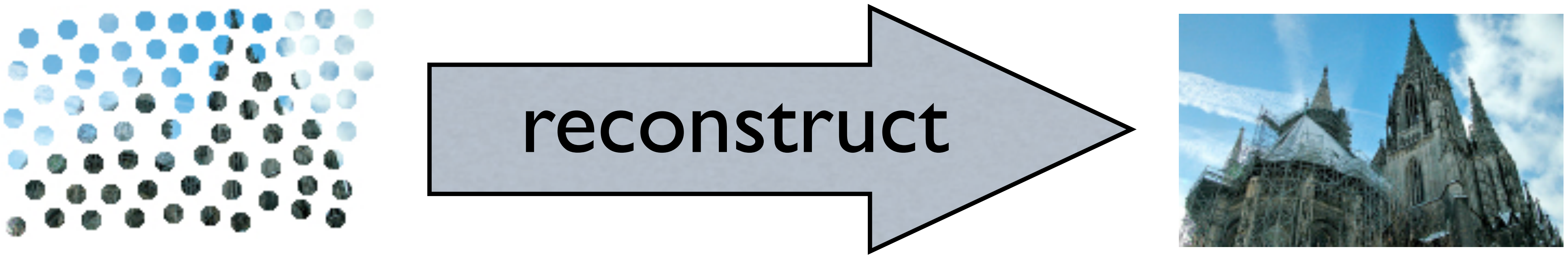
$m \times m$

Basis: $\{\boldsymbol{e}_i \boldsymbol{e}_j^\dagger\}_{i,j=1,\ldots,m}$

$\mathrm{rank}[M] = r$ specified by $\sim rm$ parameters

$\mathrm{coh}[M] = \nu$ "typical entry contains sufficient information about M"

Candes, Tao, arXiv:0903.1476



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$\mathrm{rank}[M] = r$ specified by $\sim rm$ parameters

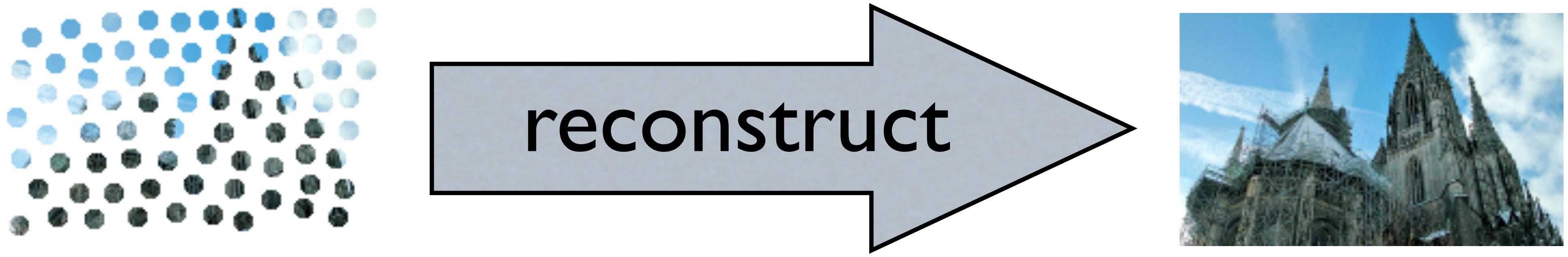
$\mathrm{coh}[M] = \nu$ “typical entry contains sufficient information about M“

sample size

$\geq C m r \nu^4 \log^2(m)$

perfect reconstruction with probability $1 - \frac{1}{m^3}$

Candes, Tao, arXiv:0903.1476



$= M$

$m \times m$

Basis: $\hat{P}_k = \bigotimes_{i=1}^{N} \hat{\sigma}_i^{\alpha_i}$

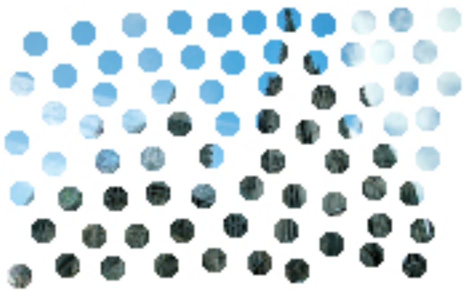
$\mathrm{rank}[M] = r$ specified by $\sim rm$ parameters

sample size

$\geq Cmr(\beta + 1)\log(m)$

perfect reconstruction
with probability $1 - \mathrm{e}^{-\beta}$

Gross, IEEE Trans. Inf. Th., **57**, 1548 (2011)



reconstruct

$= M$

$m \times m$

$$\text{minimize } \operatorname{tr}[|A|]$$
$$\text{such that } A_{i,j} = M_{i,j} \text{ for all } (i,j) \in \Omega$$

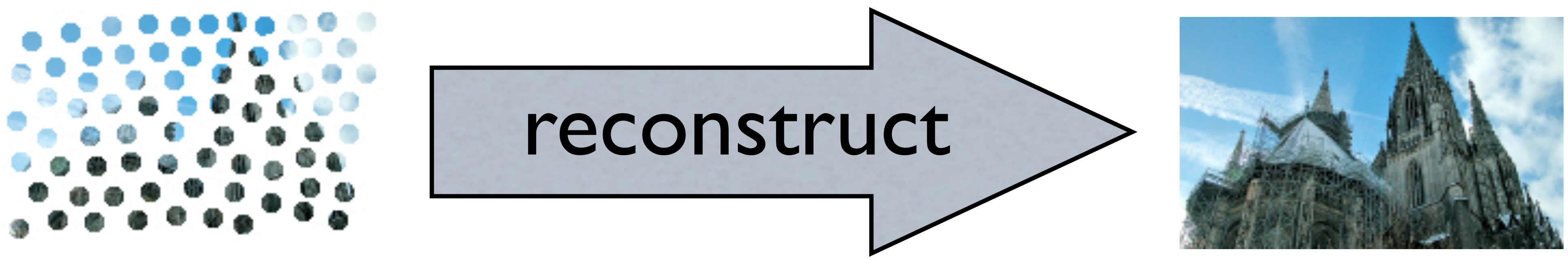
sample size

$$\geq Cmr(\beta + 1)\log(m)$$

solution is unique and equal to M with probability $1 - \mathrm{e}^{-\beta}$

Gross, IEEE Trans. Inf. Th., **57**, 1548 (2011)

Singular Value Thresholding



$= M$

$m \times m$

$$\text{minimize } \operatorname{tr}[|A|]$$
$$\text{such that } A_{i,j} = M_{i,j} \text{ for all } (i,j) \in \Omega$$

Initialize Y_0 (e.g., by zero matrix), proceed inductively

- Singular value decomposition of $2^N \times 2^N$ matrix
- Thresholding $X_n = U \max\{0, \Sigma - \mathbb{1}\tau\} V^\dagger$
- $Y_n = Y_{n-1} + \delta_n P_\Omega (M - X_n)$

Converges provably to solution for sufficiently small δ_n and $\tau \gg 1$

J-F. Cai, E.J. Candes, and Z. Shen, arXiv:0810.3286

- Some schemes
  - Permutationally invariant tomography

    Toth, Wieczorek, Gross, Krischek, Schwemmer, Weinfurter, PRL **105**, 250403 (2010)
  - Evidence procedure

    J. Rau, PRA **82**, 012104 (2010)
  - Compressed sensing

    Gross, IEEE Trans. Inf. Th., **57**, 1548 (2011)

- Efficient state representation
  - and using it for tomography

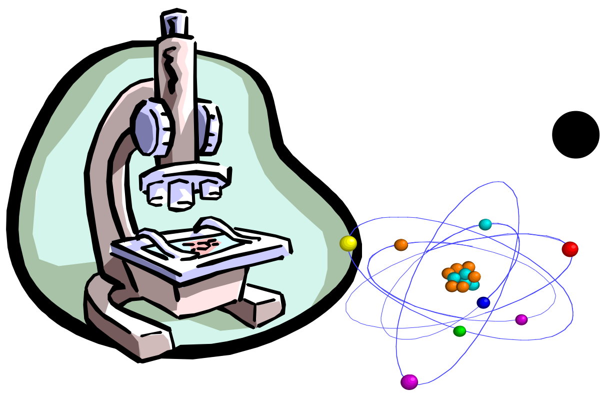
    Cramer, Plenio, Flammia, Somma, Gross, Bartlett, Landon-Cardinal, Poulin, Liu, Nat. Commun. **1**, 149 (2010)

$$\hat{\varrho} \propto \sum_k \langle \hat{P}_k \rangle \hat{P}_k$$

$$\hat{P}_k = \bigotimes_{i=1}^{N} \hat{\sigma}_i^{\alpha_i}$$

4^N observables

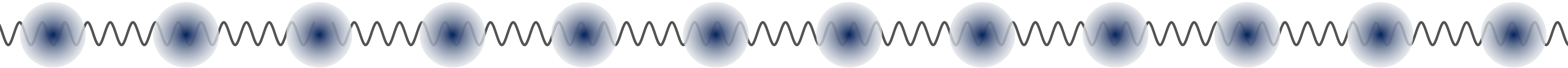
- Possible for few-particle systems
- Infeasible for large systems



- Number and accuracy of measurements

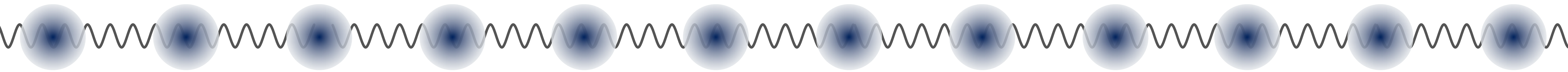
- Find compatible state
- Storage space

efficient description needed



- One-dimensional geometry
- “Local correlations stronger than those between distant subsystems”

Efficient state representation: Matrix product states



- One-dimensional geometry
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Matrix product states

$$|\psi\rangle = \sum_{s_1,\ldots,s_N} A_1[s_1]\cdots A_N[s_N]|s_1\cdots s_N\rangle$$

Fannes, Nachtergaele, Werner, Comm. Math. Phys. **144**, 443 (1992),
Schollwöck, Rev. Mod. Phys. **77**, 259 (2005),
Perez-Garcia, Verstraete, Wolf, Cirac, Quant. Inf. Comp. **7**, 401 (2007).

$$|\psi\rangle = \sum_{s_1,\ldots,s_N} A_1[s_1]\cdots A_N[s_N]|s_1\cdots s_N\rangle$$

- Efficient if dimension low; degrees of freedom $\sim N\chi^2$
- W, GHZ states: $\chi = 2$
- DMRG: Variation over MPS
- MPS approximate ground states very well
- Generic MPS is unique ground state of local Hamiltonian
- Reductions can be computed efficiently:

$$\hat{\varrho}_{j,\ldots,j+k} = \sum_{\substack{s_j,\ldots,s_{j+k}\\ s'_j,\ldots,s'_{j+k}}} \mathrm{tr}\left[A_j[s_j]\cdots A_{j+k}[s_{j+k}]A^\dagger_{j+k}[s'_{j+k}]\cdots A^\dagger_j[s'_j]\right]|s_j\cdots s_{j+k}\rangle\langle s'_j\cdots s'_{j+k}|$$

$$\mathrm{rank}[\hat{\varrho}_{j,\ldots,j+k}] \leq \chi^2$$

Fannes, Nachtergaele, Werner, Comm. Math. Phys. **144**, 443 (1992),
Schollwöck, Rev. Mod. Phys. **77**, 259 (2005),
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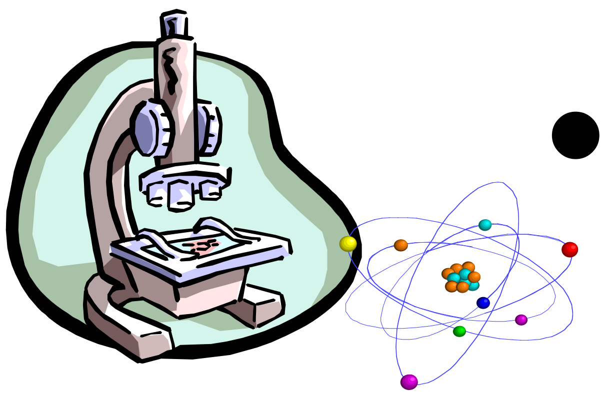
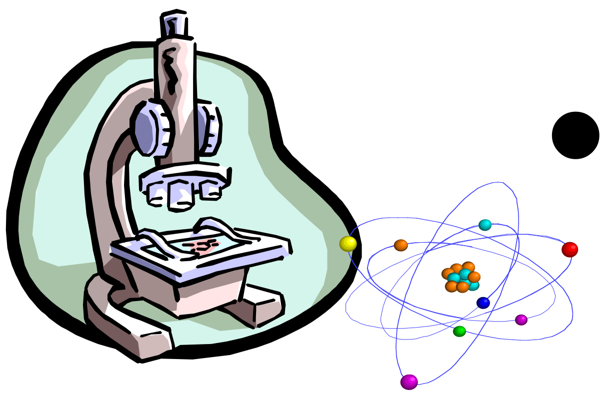
Quantum state tomography: A solution?

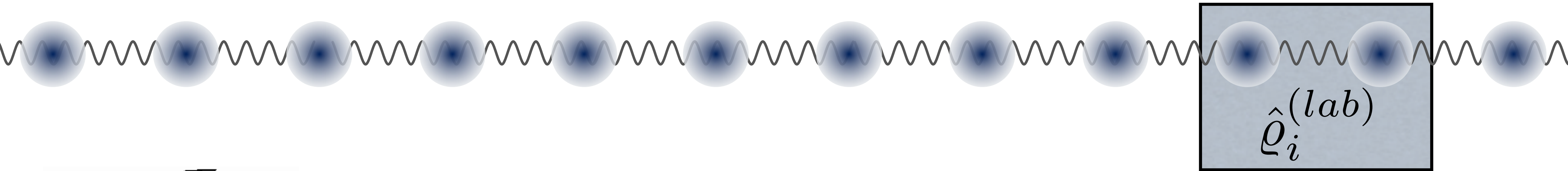


- Number and accuracy of measurements

- Find compatible state
- Storage space ✓



- Number and accuracy of measurements → Take only $\sim N$



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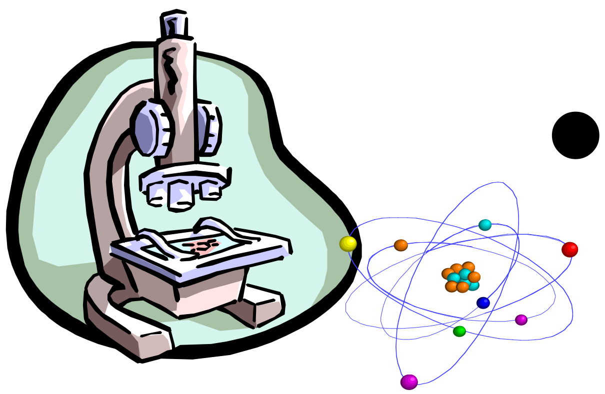
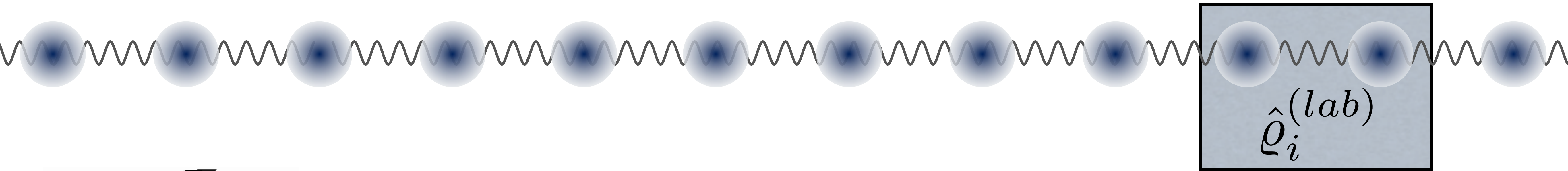
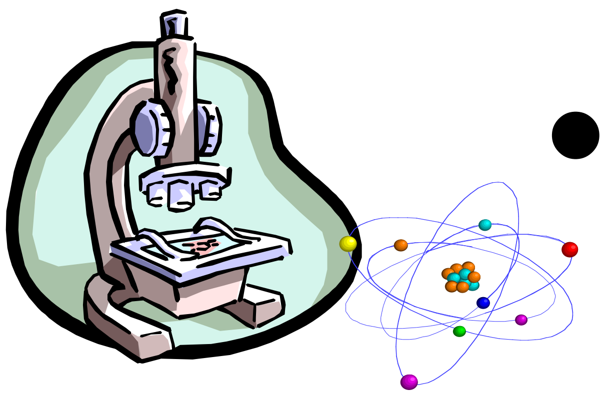


- Number and accuracy of measurements → Take only 4^2N

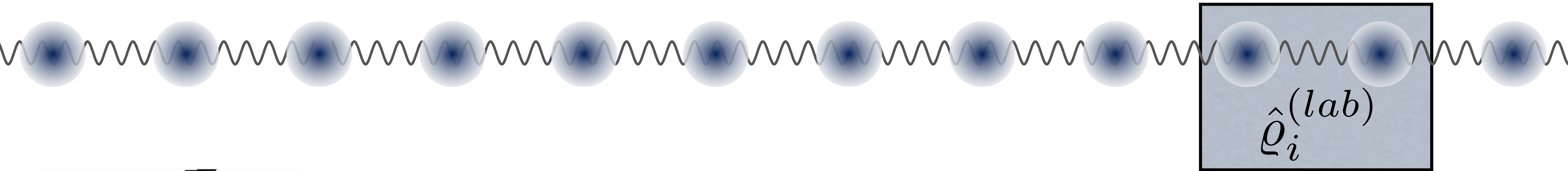


- *Promise:* $\hat{\varrho}_{lab} = |\psi\rangle\langle\psi|$ is unique ground state of local Hamiltonian $\hat{H} = \sum_i \hat{h}_i$:

Quantum state tomography: A solution?



- Number and accuracy of measurements → Take only $4^2 N$



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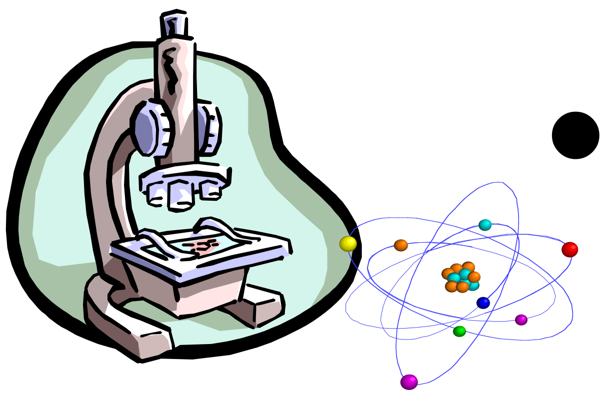
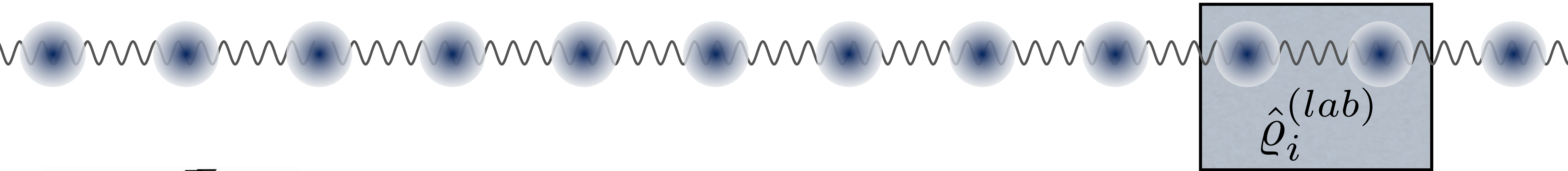
  Candidate $\hat{\varrho}_{cand} = |\phi\rangle\langle\phi|$ with
  $\hat{\varrho}_i^{(lab)} = \hat{\varrho}_i^{(cand)}$

Quantum state tomography: A solution?



- Number and accuracy of measurements → Take only $4^2 N$



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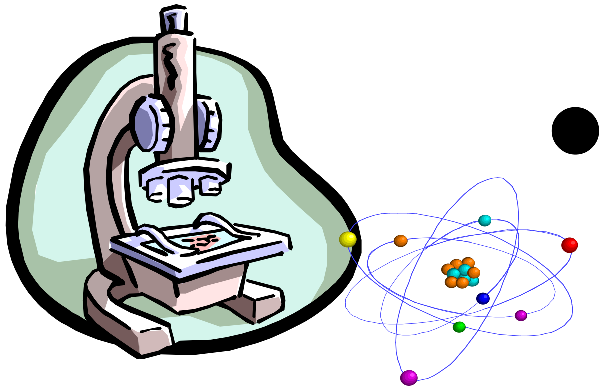
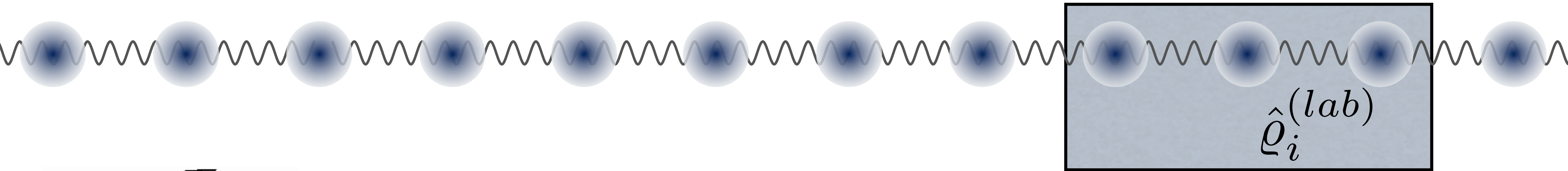


- Number and accuracy of measurements → Take only $4^3 N$



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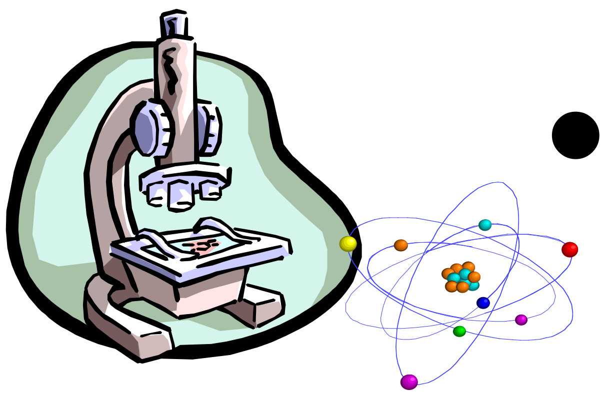
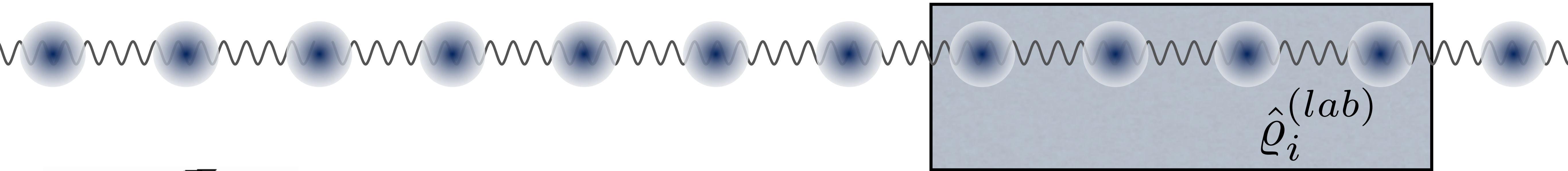
  $\hat{\varrho}_i^{(lab)} = \hat{\varrho}_i^{(cand)} \rightarrow \langle\psi|\hat{h}_i|\psi\rangle = \langle\phi|\hat{h}_i|\phi\rangle$
  $\rightarrow \langle\psi|\hat{H}|\psi\rangle = \langle\phi|\hat{H}|\phi\rangle \rightarrow |\psi\rangle = |\phi\rangle$

- General local Hamiltonians, limit



- Number and accuracy of measurements → Take only $4^4 N$



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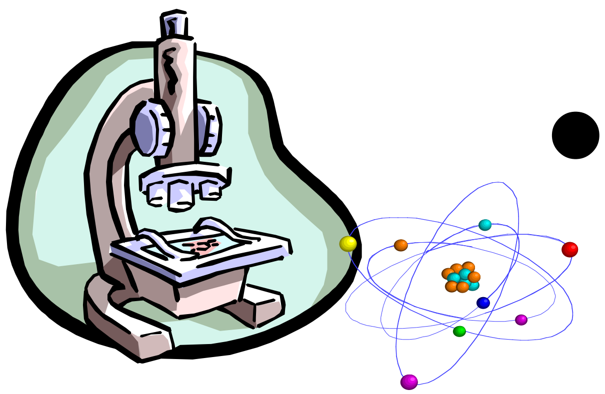
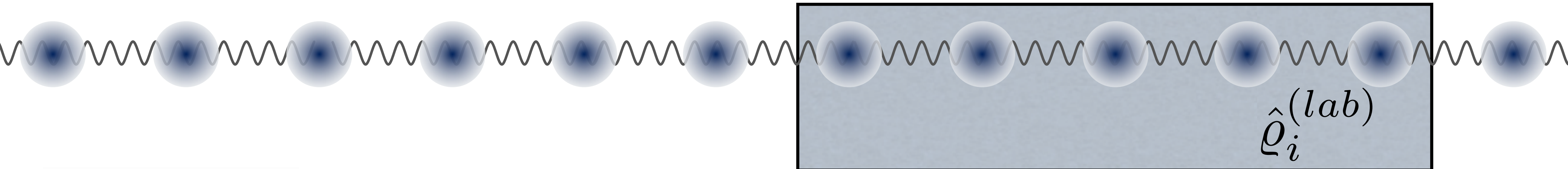
  $\hat{\varrho}_i^{(lab)} = \hat{\varrho}_i^{(cand)}$ → $\langle\psi|\hat{h}_i|\psi\rangle = \langle\phi|\hat{h}_i|\phi\rangle$

  → $\langle\psi|\hat{H}|\psi\rangle = \langle\phi|\hat{H}|\phi\rangle$ → $|\psi\rangle = |\phi\rangle$

- General local Hamiltonians, limit



- Number and accuracy of measurements → Take only $\sim N$



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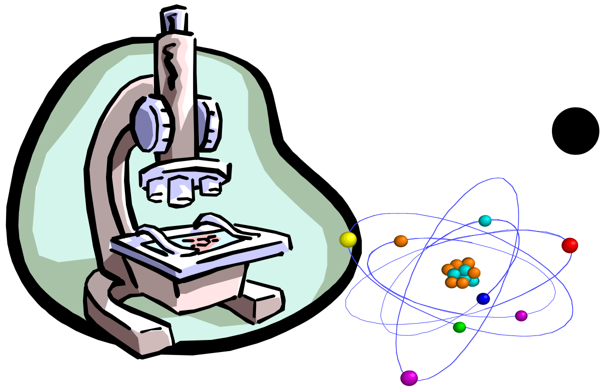
$$\rightarrow \langle\psi|\hat{H}|\psi\rangle = \langle\phi|\hat{H}|\phi\rangle \rightarrow |\psi\rangle = |\phi\rangle$$

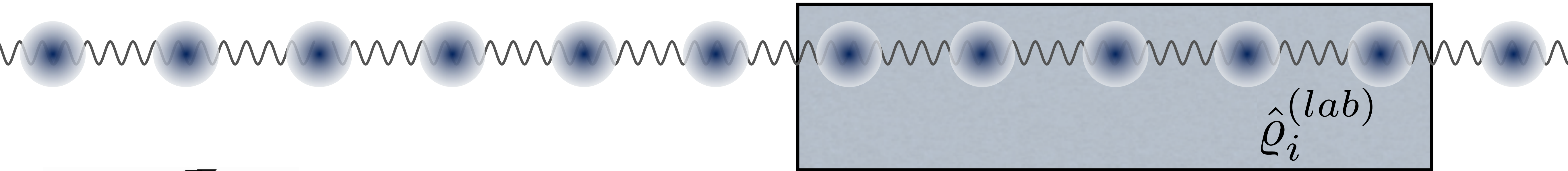
- General local Hamiltonians, limit

Quantum state tomography: A solution?



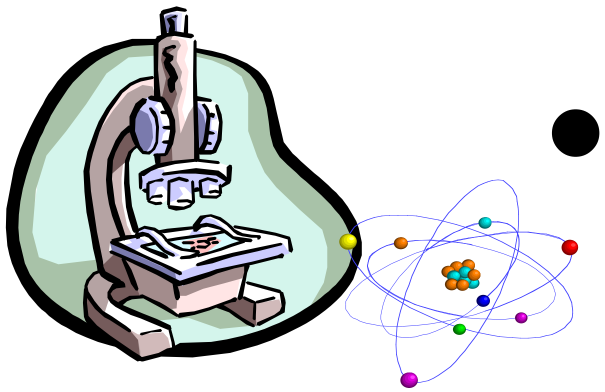
- Number and accuracy of measurements ✓

Take only $\sim N$



- Find compatible state
- Storage space ✓

Quantum state tomography: A solution?



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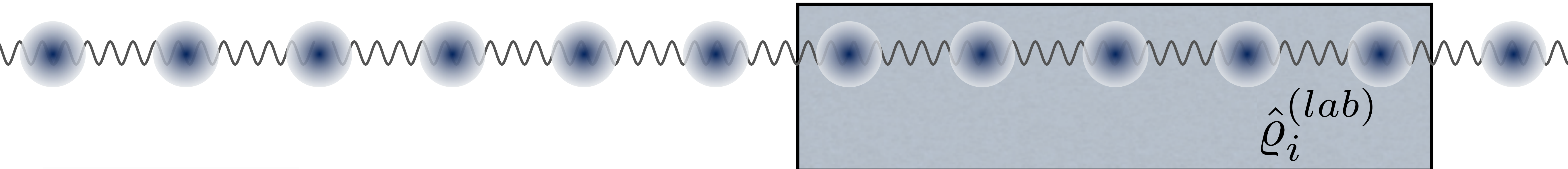
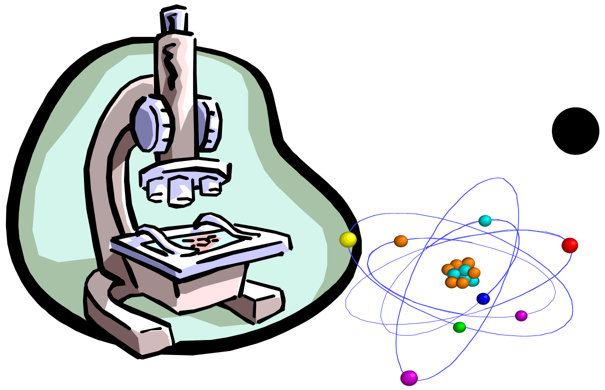


- Find compatible state
- Storage space ✓

$$\left\| \hat{\varrho}_i^{(lab)} - \hat{\varrho}_i^{(est)} \right\|_{\mathrm{tr}} \leq \epsilon_i$$

$|\psi\rangle$ **and** $1 - \langle\psi|\hat{\varrho}_{lab}|\psi\rangle \leq \frac{\sum_i (\epsilon_i + \mathrm{tr}[\hat{h}_i \hat{\varrho}_i^{(est)}])}{\Delta E}$



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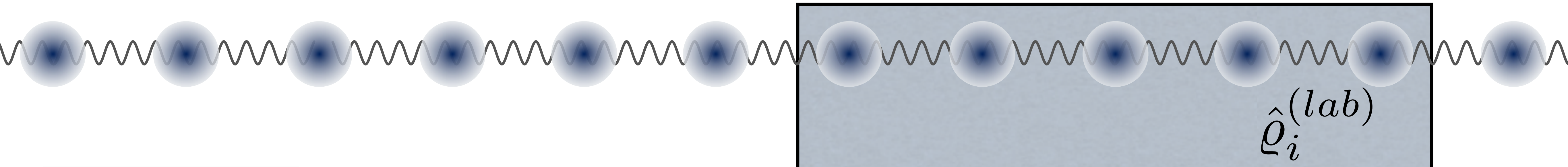

- Find compatible state
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- Find parent Hamiltonian, ensure uniqueness of ground state and compute gap efficiently

$$\left\| \hat{\varrho}_i^{(lab)} - \hat{\varrho}_i^{(est)} \right\|_{\mathrm{tr}} \leq \epsilon_i$$

$$|\psi\rangle \text{ and } 1 - \langle\psi|\hat{\varrho}_{lab}|\psi\rangle \leq \frac{\sum_i (\epsilon_i + \mathrm{tr}[\hat{h}_i \hat{\varrho}_i^{(est)}])}{\Delta E}$$

$$|\psi\rangle = \sum_{s_1,\ldots,s_N} A_1[s_1]\cdots A_N[s_N]|s_1\cdots s_N\rangle$$

If $\left\{A_{i+1}[s_1]\cdots A_{i+k}[s_k]\,\middle|\, s_j = 1,\ldots,d_j\right\}$ spans $\mathbb{C}^{\chi_i\times\chi_{i+k}}$

→ $|\psi\rangle$ is unique ground state of its parent Hamiltonian

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$$\Delta E = \max\left\{\lambda \,\middle|\, \hat{H}(\hat{H}-\lambda)\geq 0\right\}$$

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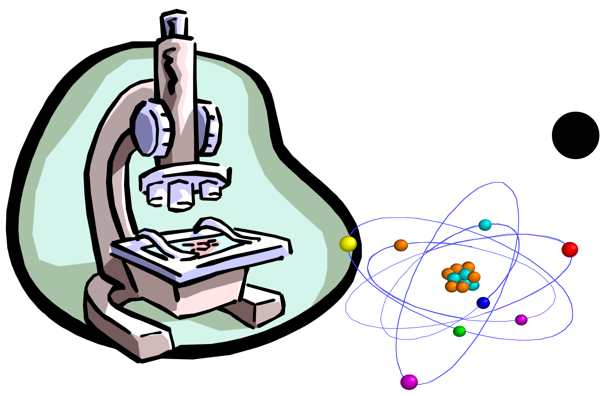
→ $|\psi\rangle$ is unique ground state of its parent Hamiltonian

$$\Delta E = \max\left\{\lambda \,\middle|\, \hat{H}(\hat{H}-\lambda)\geq 0\right\}$$

$$\hat{H}^2 = \hat{H} + \sum_{\substack{i,j\\ i\neq j}} \hat{h}_i\hat{h}_j \geq \hat{H} + \sum_{\substack{i,j\\ i\neq j\\ overlap}} \frac{\hat{h}_i\hat{h}_j + \hat{h}_j\hat{h}_i}{2} \geq (1-\gamma)\hat{H}$$

→ $\Delta E \geq 1-\gamma$

γ: simple function of smallest non-zero eigenvalue of $\hat{h}_i + \hat{h}_j$



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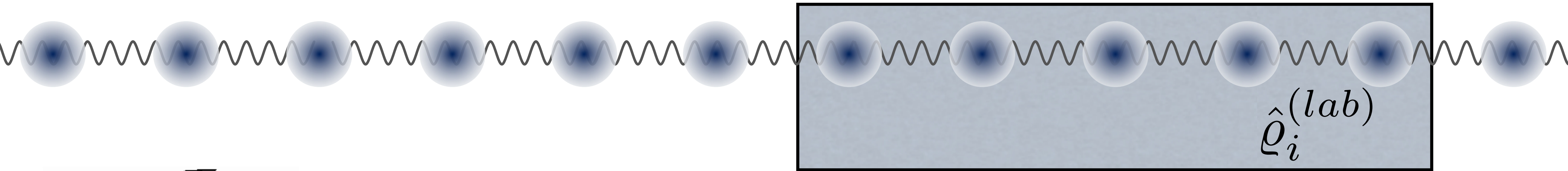
Take only $\sim N$



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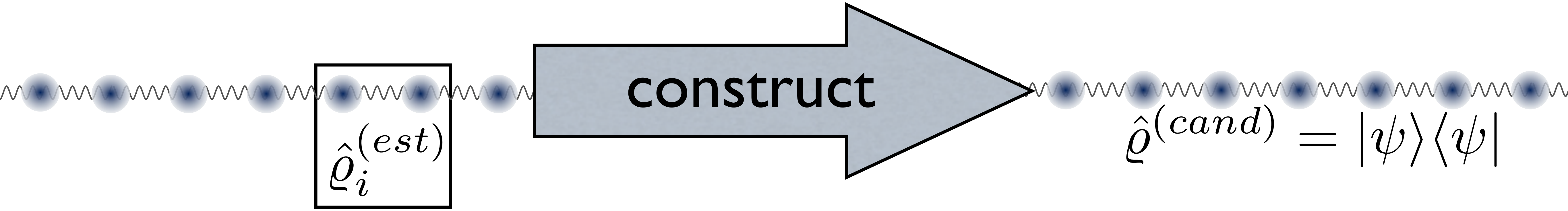
- Find compatible state, i.e., $|\psi\rangle$ such that $\hat{\varrho}_i^{(est)} = \hat{\varrho}_i^{(cand)}$



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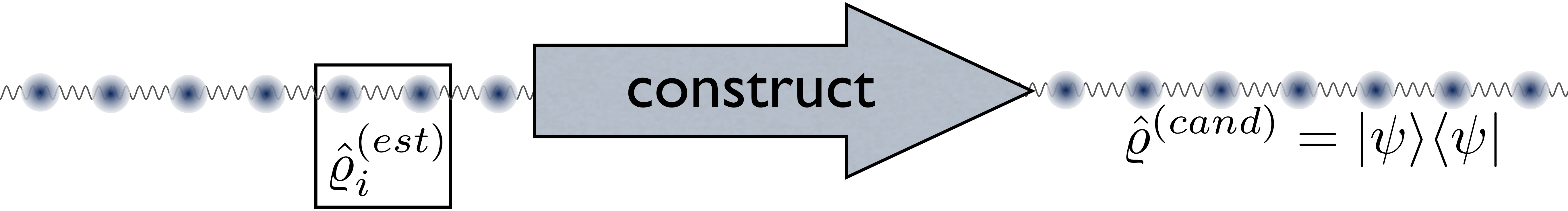
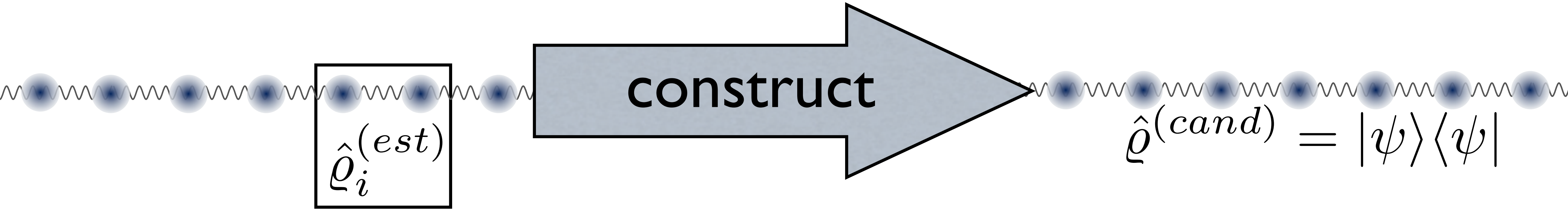


- Directly minimize $\sum_i \|\hat{\varrho}_i^{(est)} - \hat{\varrho}_i^{(cand)}\|$
- Find ground state of Hamiltonian constructed from local estimates

Quantum state tomography: A solution?

- Find compatible state, i.e., $|\psi\rangle$ such that $\hat{\varrho}_i^{(est)} = \hat{\varrho}_i^{(cand)}$



measured Entries:

$$\Omega = \{k : \ \hat{P}_k = \mathbb{1} \otimes \hat{\sigma}_i^{\alpha_i} \otimes \hat{\sigma}_{i+1}^{\alpha_{i+1}} \otimes \mathbb{1}\}$$

Entries:

$$p_k = \mathrm{tr}[\hat{P}_k |\psi\rangle\langle\psi|]$$

$$\hat{P}_k = \bigotimes_{i=1}^{N} \hat{\sigma}_i^{\alpha_i}$$

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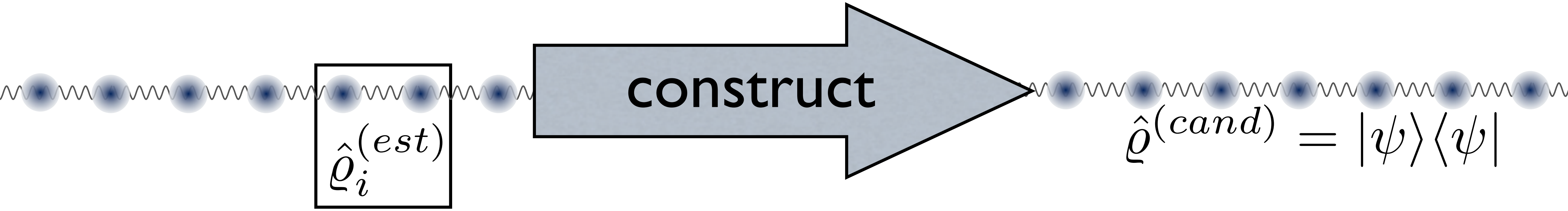

$\hat{\varrho}_i^{(est)}$

$\hat{\varrho}^{(cand)} = |\psi\rangle\langle\psi|$

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$$p_k = \mathrm{tr}[\hat{P}_k |\psi\rangle\langle\psi|]$$

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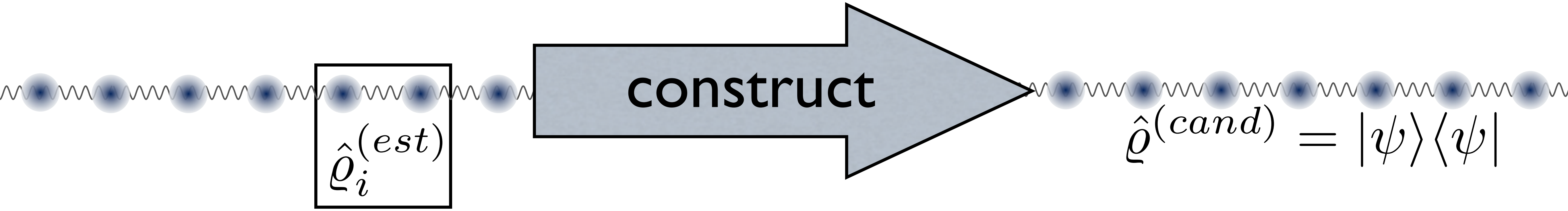
- Directly minimize $\sum_i \|\hat{\varrho}_i^{(est)} - \hat{\varrho}_i^{(cand)}\|$
- Find ground state of Hamiltonian constructed from local estimates
- Singular value thresholding

- Find compatible state, i.e., $|\psi\rangle$ such that $\hat{\varrho}_i^{(est)} = \hat{\varrho}_i^{(cand)}$



measured Entries:

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Initialize Y_0 (e.g., by zero matrix), proceed inductively

- Singular value decomposition of $2^N \times 2^N$ matrix
- Thresholding $X_n = U \max\{0, \Sigma - \mathbb{1}\tau\} V^\dagger$
- $Y_n = Y_{n-1} + \delta_n \sum_{k \in \Omega} \frac{p_k - \mathrm{tr}[X_n \hat{P}_k]}{2^N} \hat{P}_k$

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Instead of thresholding: Keep only largest singular value

Initialize Y_0 (e.g., by zero matrix), proceed inductively

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$$= \sum_{k\in\Omega} a_k \hat{P}_k, \quad a_k \in \mathbb{R}$$

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local "Hamiltonian", find ground state, compute expectation values efficiently

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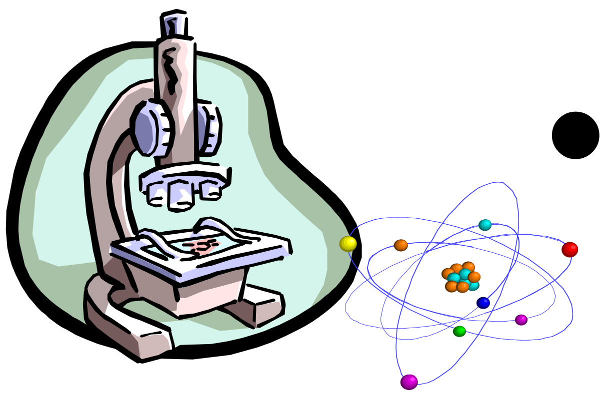
local "Hamiltonian", find ground state, compute expectation values efficiently

DMRG, MPS methods

M. Fannes, B. Nachtergaele, and R.F. Werner, Comm. Math. Phys. **144**, 443 (1992),
U. Schollwöck, Rev. Mod. Phys. **77**, 259 (2005),
D. Perez-Garcia, F. Verstraete, M.M. Wolf, and J.I. Cirac, Quant. Inf. Comp. **7**, 401 (2007).

Quantum state tomography: A solution?



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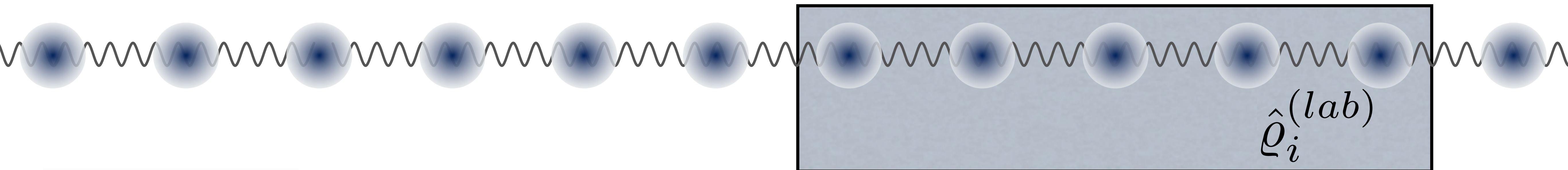

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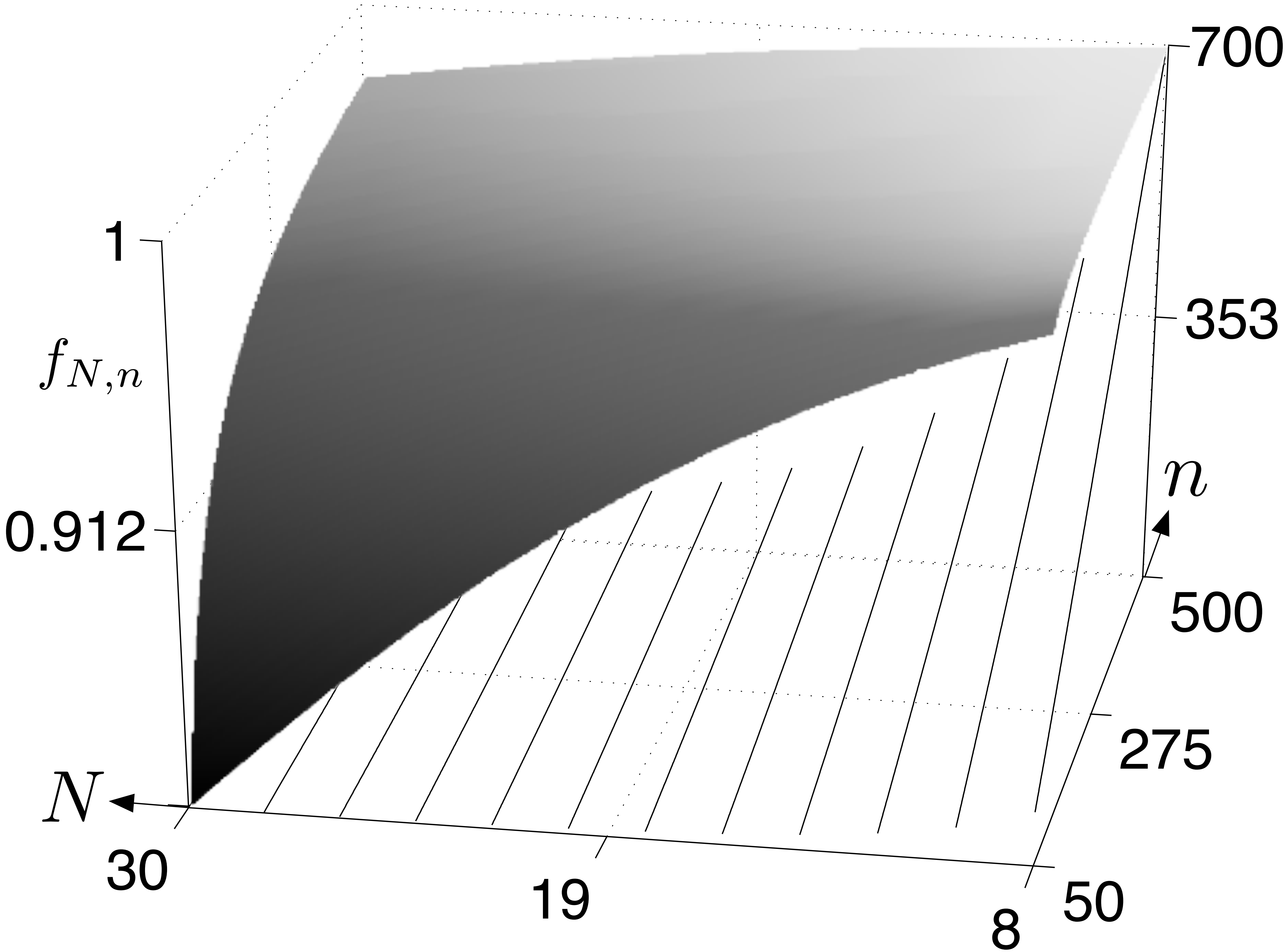
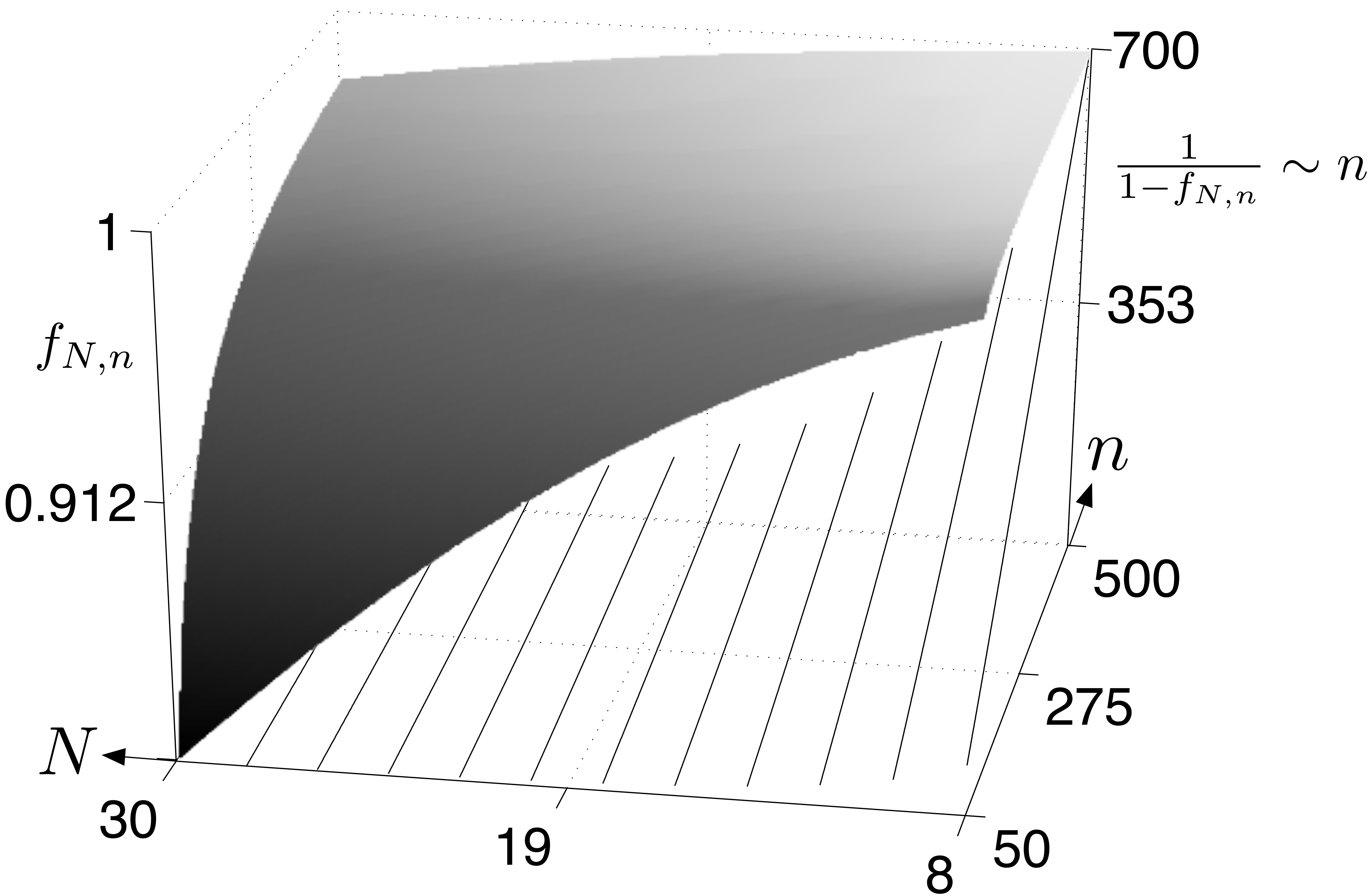
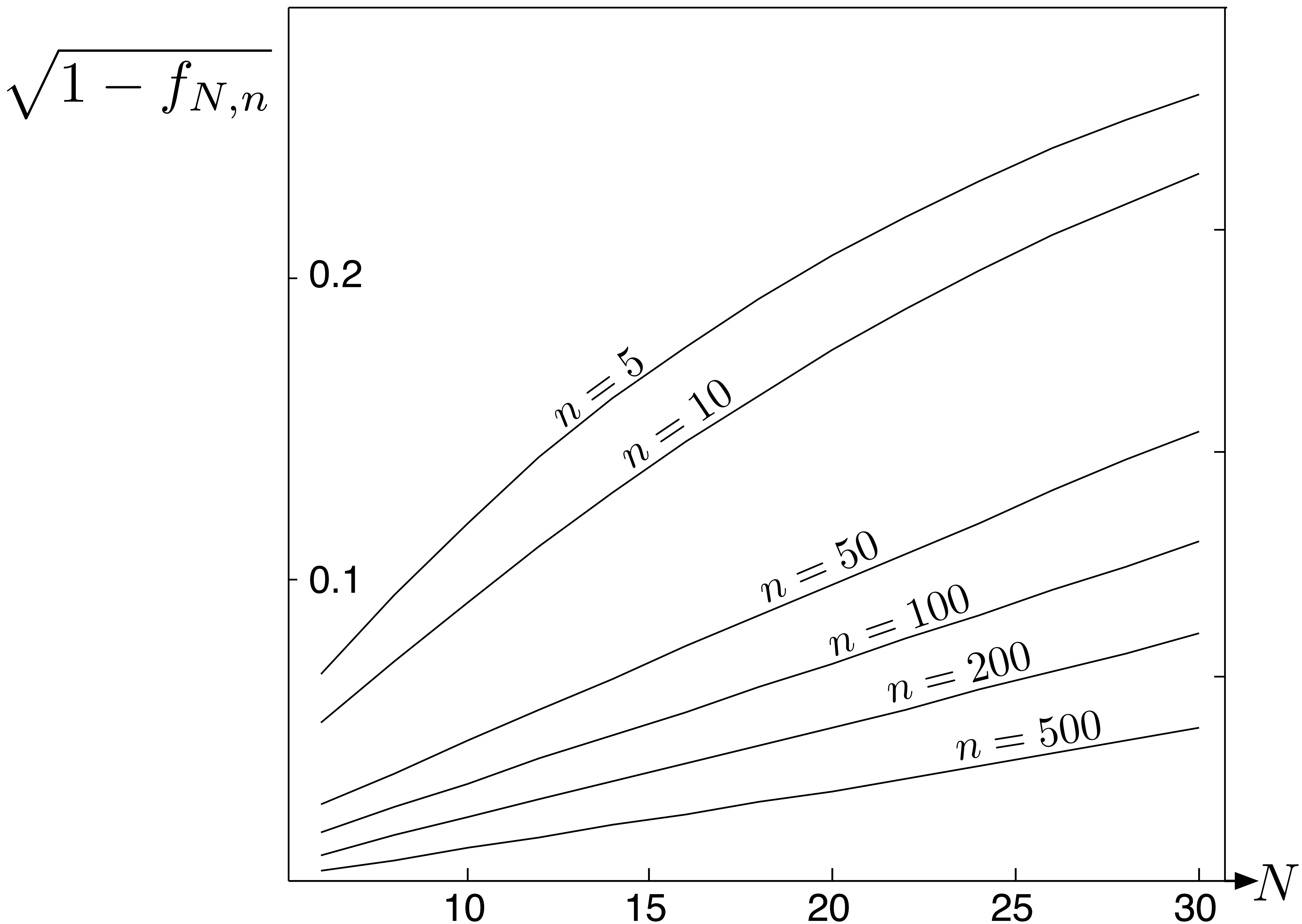
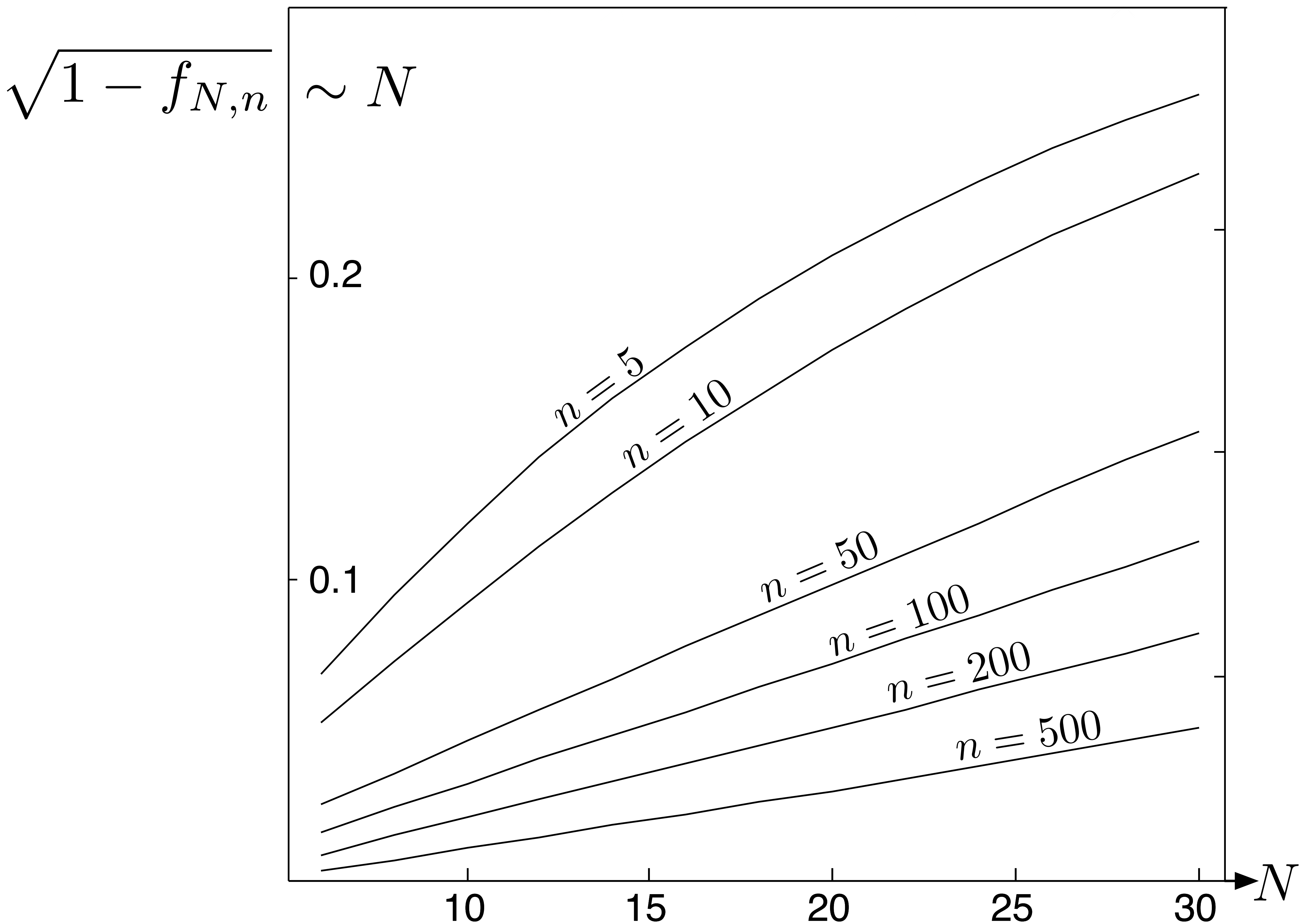
Ground state critical Ising model $\hat{H} = -\sum_{i=1}^{N-1} \hat{\sigma}_i^x \hat{\sigma}_{i+1}^x - \sum_{i=1}^{N} \hat{\sigma}_i^z$

- "Measure" all $\hat{\varrho}_{i,i+1}$
- Completely determines ground state
- Compute fidelity $f_{N,n} = |\langle gs|X_n\rangle|^2$


700
1
353
$f_{N,n}$
n
0.912
500
275
N
30
19
8
50


700
$\frac{1}{1-f_{N,n}} \sim n$
1
353
$f_{N,n}$
n
0.912
500
275
N
30
19
8
50


$\sqrt{1 - f_{N,n}}$
0.2
0.1
$n = 5$
$n = 10$
$n = 50$
$n = 100$
$n = 200$
$n = 500$
10
15
20
25
30
N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$$\hat{H} = \sum_{i=1}^{N-1} \hat{r}_i^{(i)} \hat{r}_{i+1}^{(i)}$$

hermitian, real and imaginary part of entries uniformly from $[-1, 1]$, 1000 realizations for each N, $n = 5$

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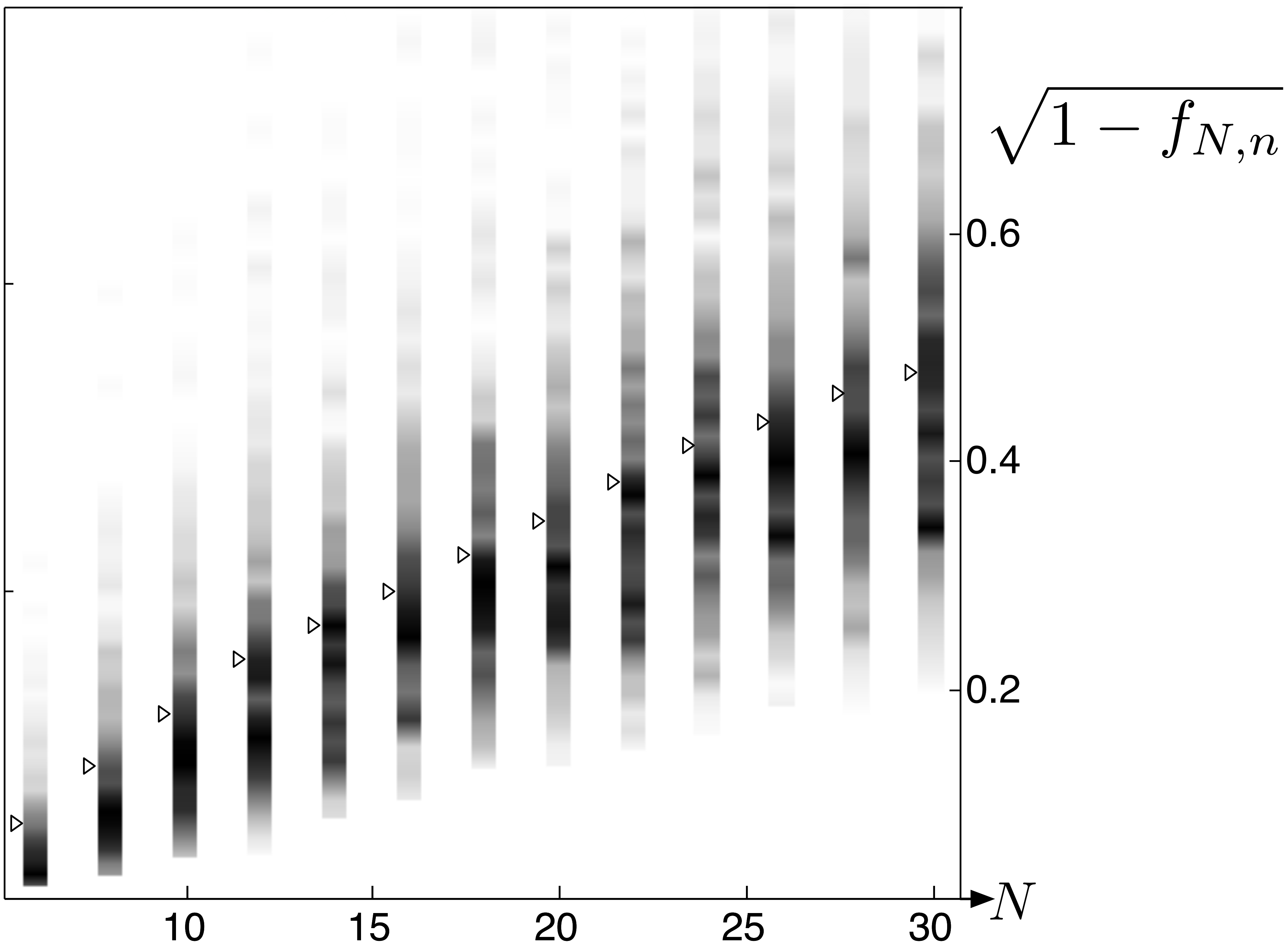
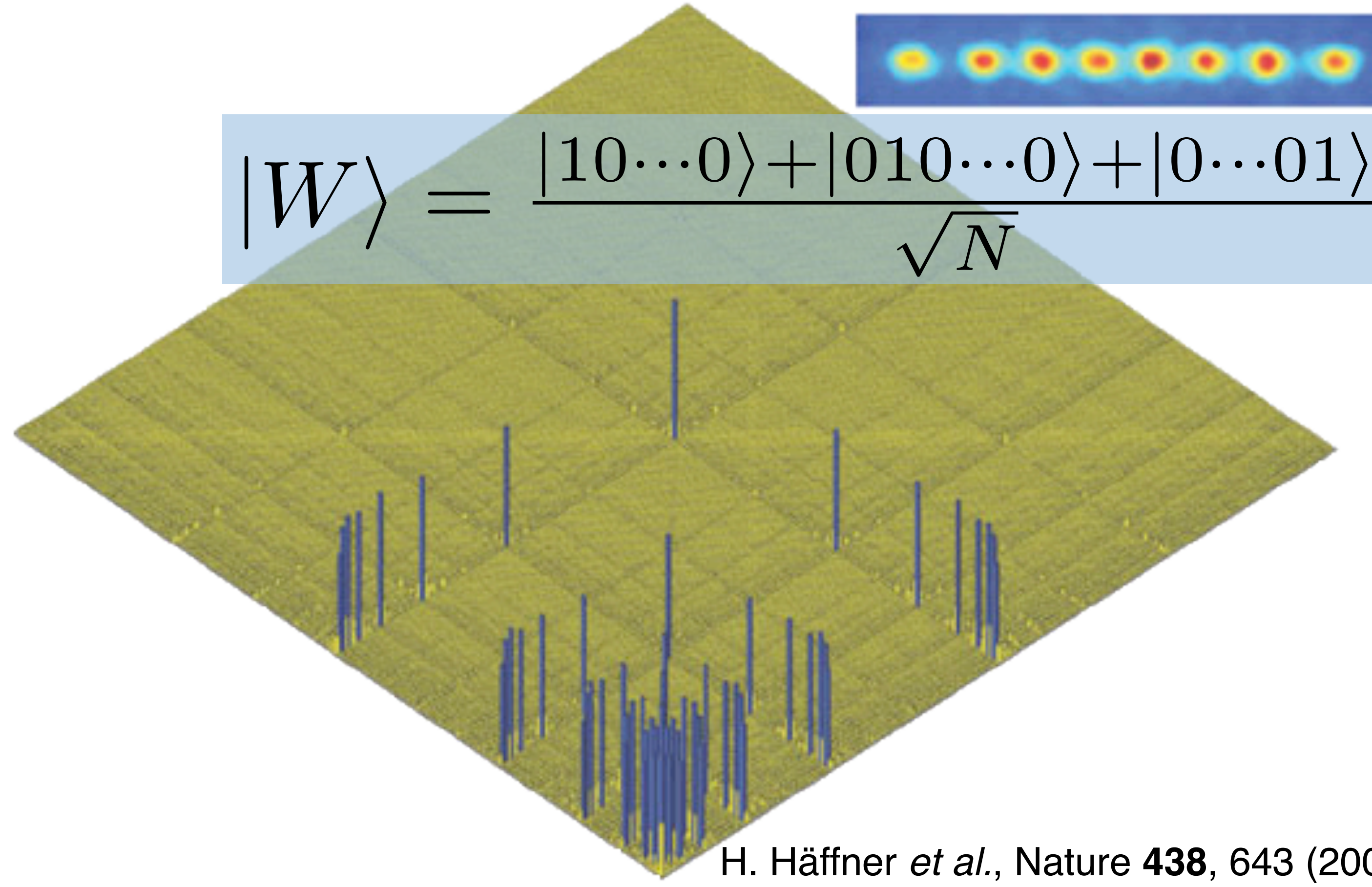


MPS-SVT: Numerical experiments



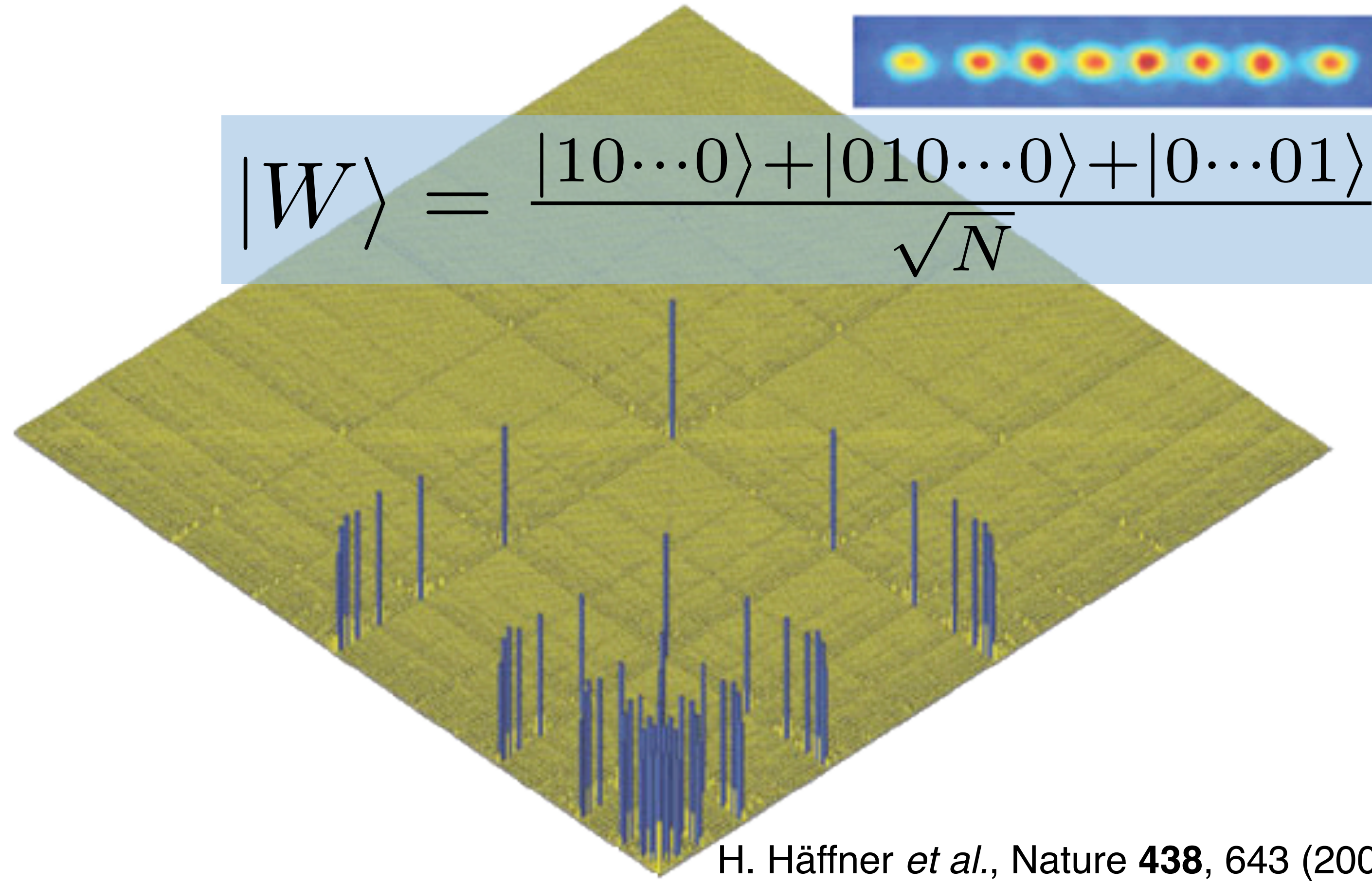
$$|W\rangle = \frac{|10\cdots0\rangle + |010\cdots0\rangle + |0\cdots01\rangle}{\sqrt{N}}$$

H. Häffner *et al.*, Nature **438**, 643 (2005).

$$p_k = \langle W | \hat{P}_k | W \rangle + r$$

Gaussian, zero mean

100 realizations for each N



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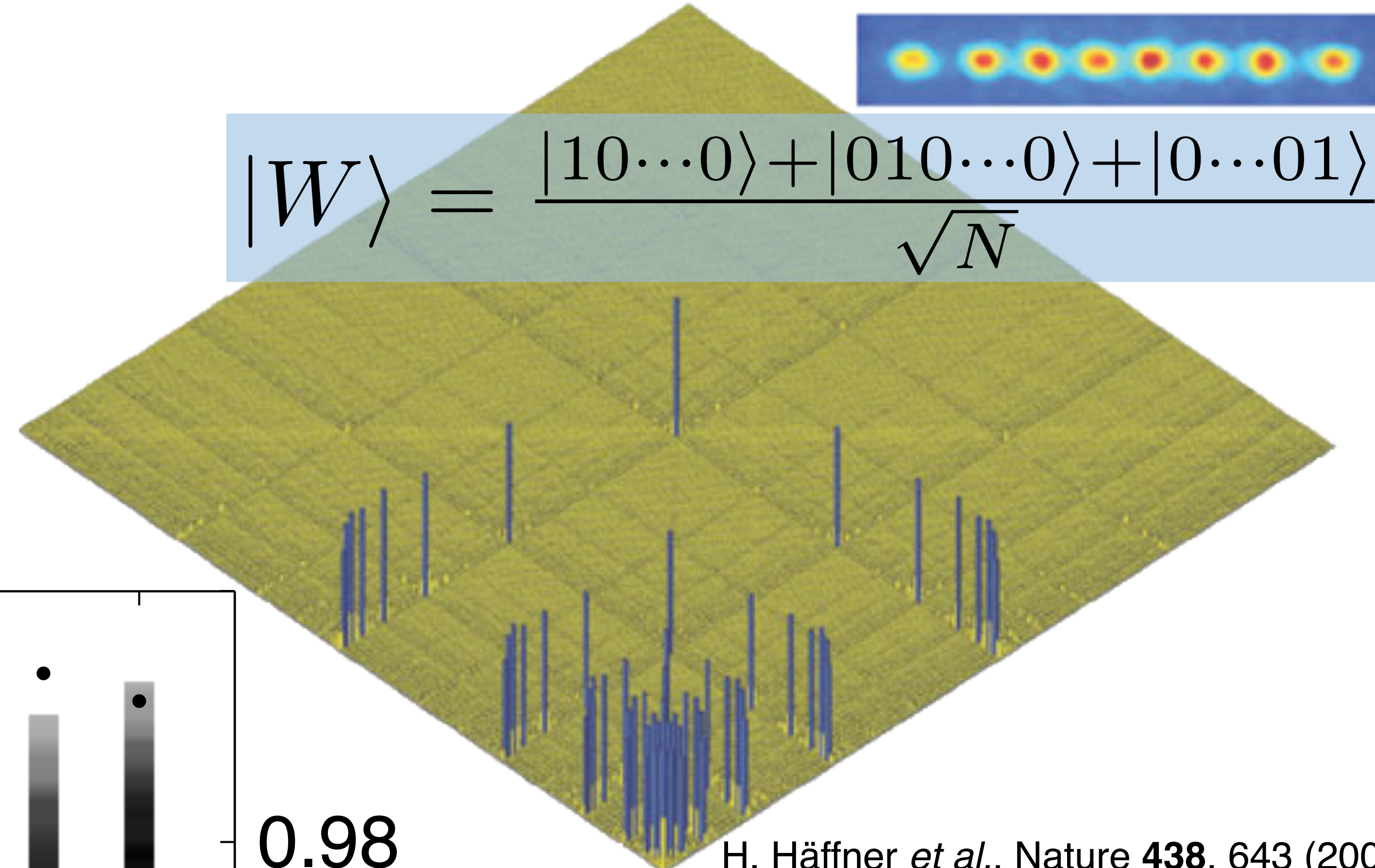

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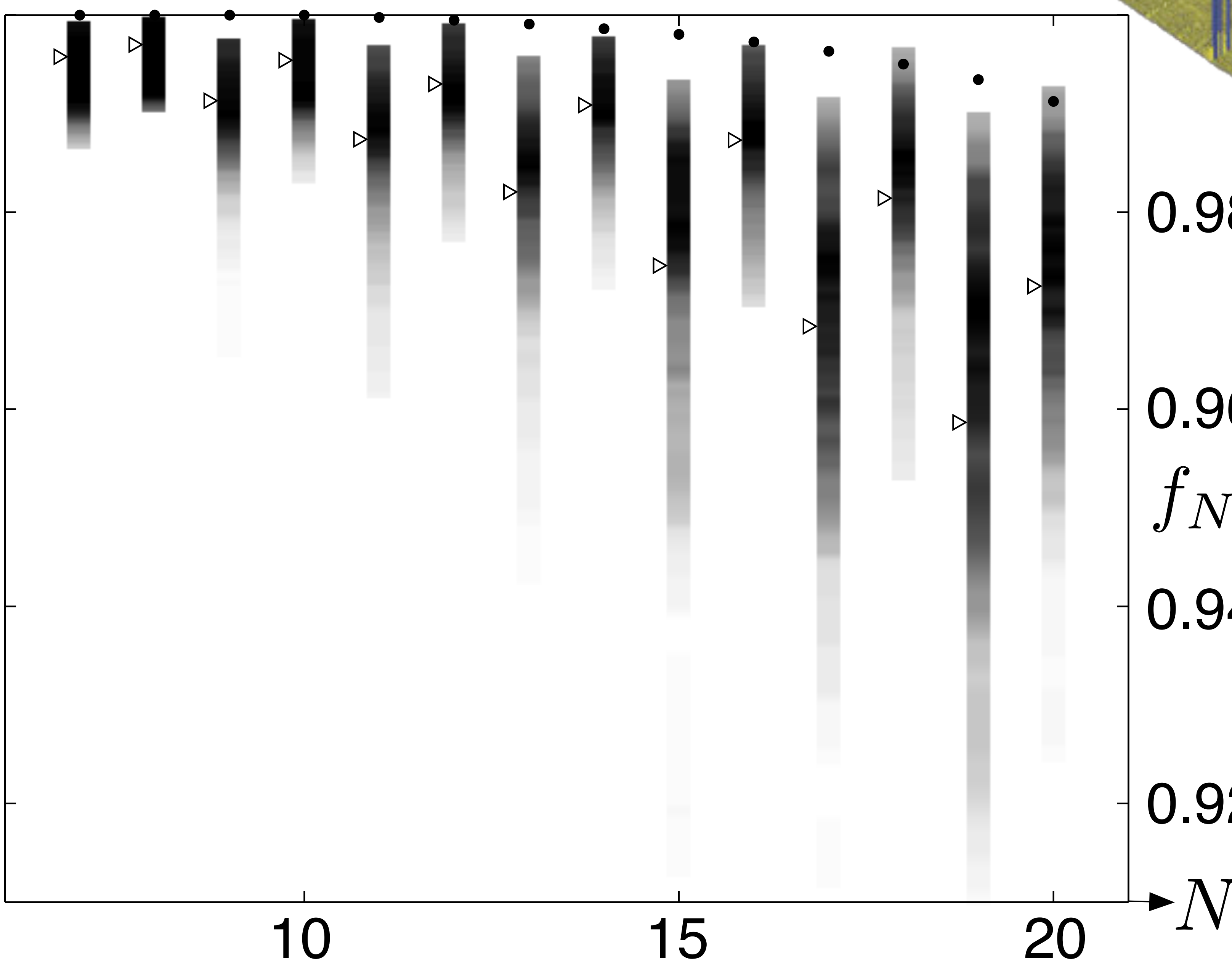


$$f_{N,n} = |\langle W|X_n\rangle|^2$$

$$n = 4000$$

$\sigma = 0.01$ (odd N)

$\sigma = 0.005$ (even N)

- Some schemes
  - Permutationally invariant tomography

    Toth, Wieczorek, Gross, Krischek, Schwemmer, Weinfurter, PRL **105**, 250403 (2010)
  - Evidence procedure

    J. Rau, PRA **82**, 012104 (2010)
  - Compressed sensing

    Gross, IEEE Trans. Inf. Th., **57**, 1548 (2011)

- Efficient state representation
  - and using it for tomography

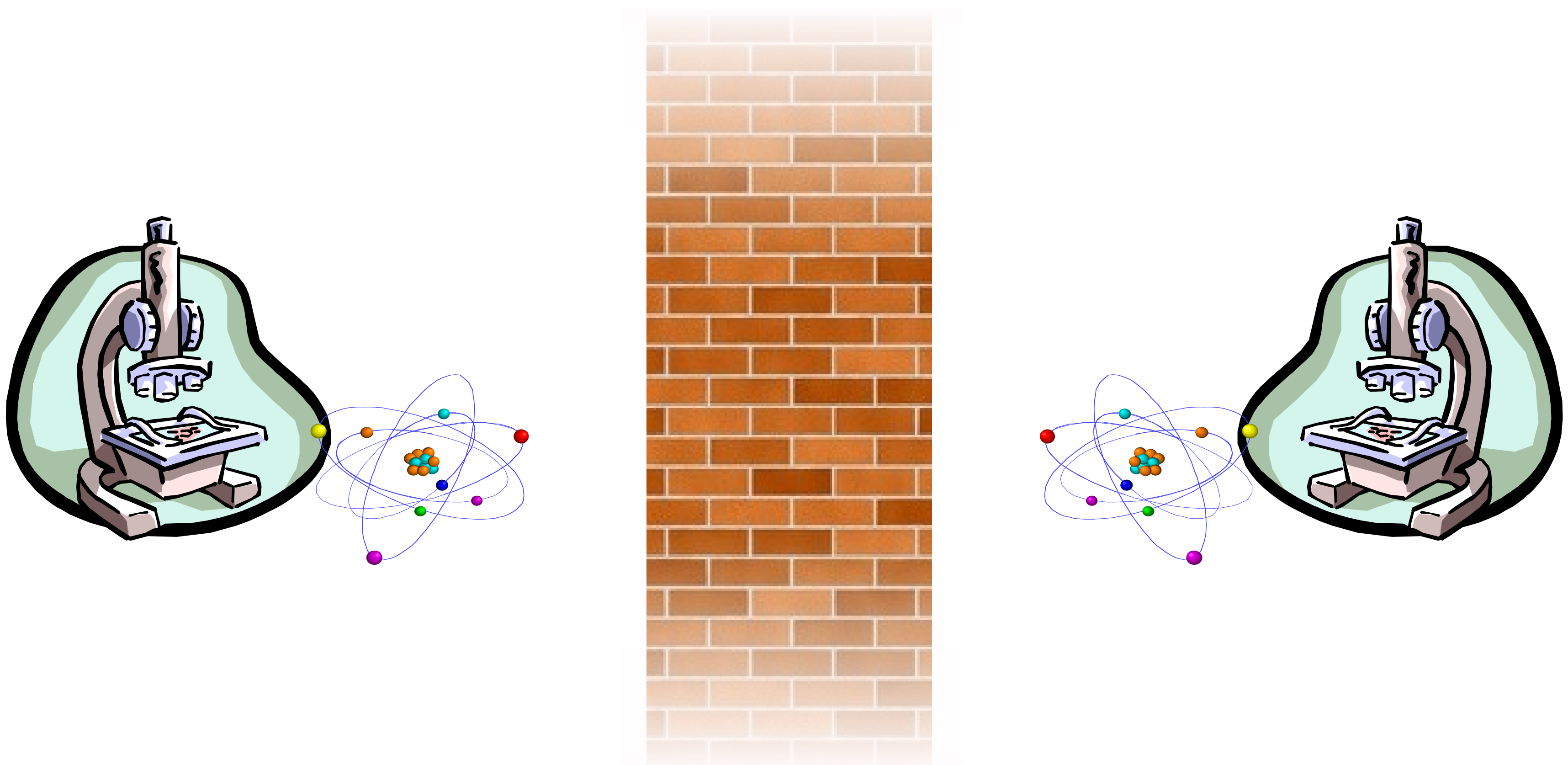
    Cramer, Plenio, Flammia, Somma, Gross, Bartlett, Landon-Cardinal, Poulin, Liu, Nat. Commun. **1**, 149 (2010)

Measuring entanglement in many-body systems

Cramer, Plenio, Wunderlich, PRL **106**, 020401 (2011)

- Rarely, experiment has access to full state tomography
- Can we quantify entanglement under minimal assumptions?
  - If we know Hamiltonian, knowledge of temperature is enough, but how do we determine Hamiltonian? Even more costly than state tomography.
  - Would like to verify existence of entanglement quantitatively in experimentally simple ways and without unspoken assumptions.

Quantifying entanglement with routine measurements: Of optimists and pessimists



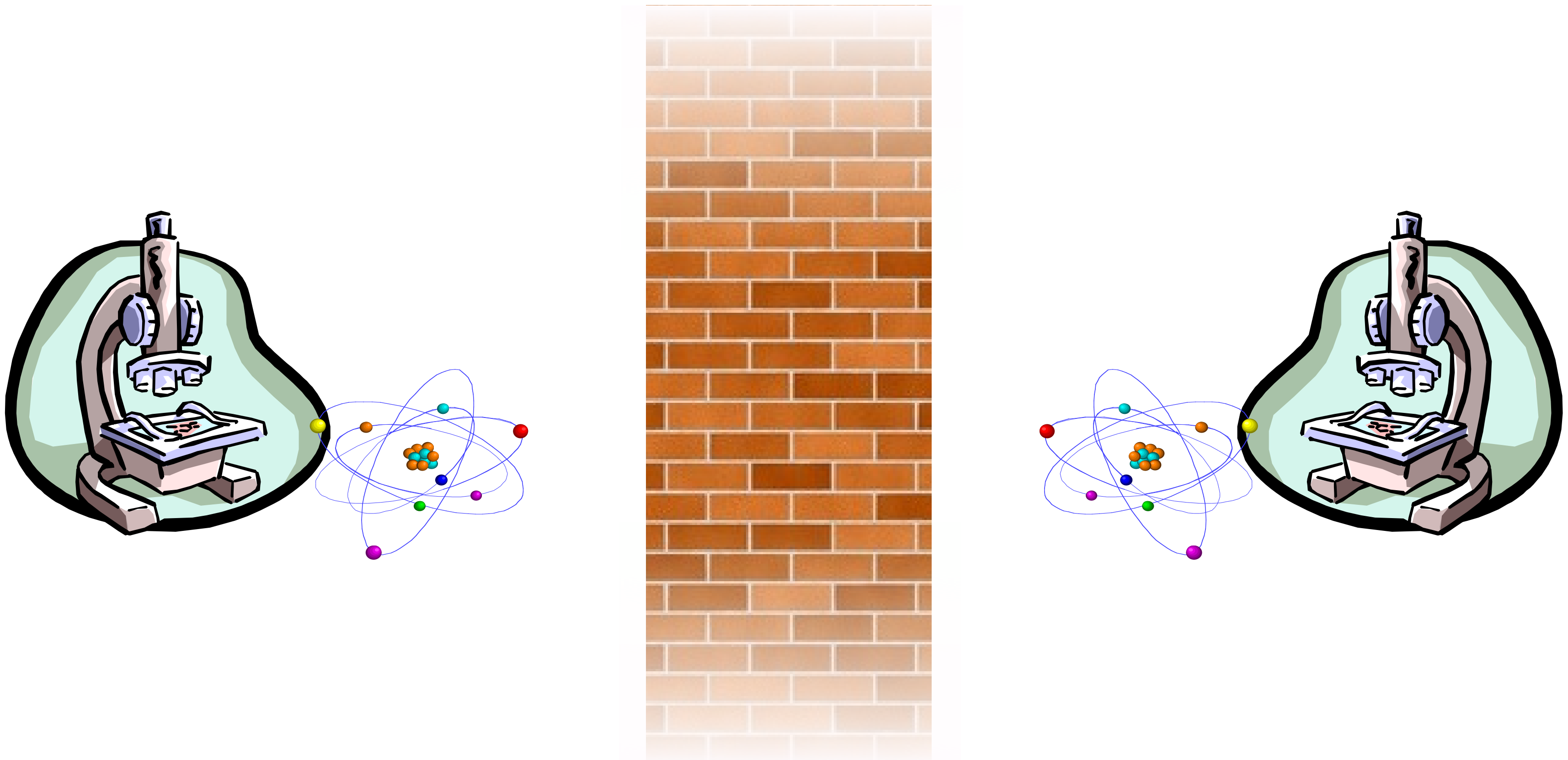
Measure: $C_{zz}(\hat{\varrho}) = \mathrm{tr}\left[\hat{\varrho}\left(\hat{\sigma}_A^z \otimes \hat{\sigma}_B^z\right)\right] - \mathrm{tr}\left[\hat{\varrho}\hat{\sigma}_A^z\right]\mathrm{tr}\left[\hat{\varrho}\hat{\sigma}_B^z\right]$

Observe: $C_{zz}(\hat{\varrho}) = -1$

Optimist: $|\psi\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$

Realist: Need to know more!

Pessimist: $\hat{\varrho} = \frac{|01\rangle\langle 01| + |10\rangle\langle 10|}{2}$



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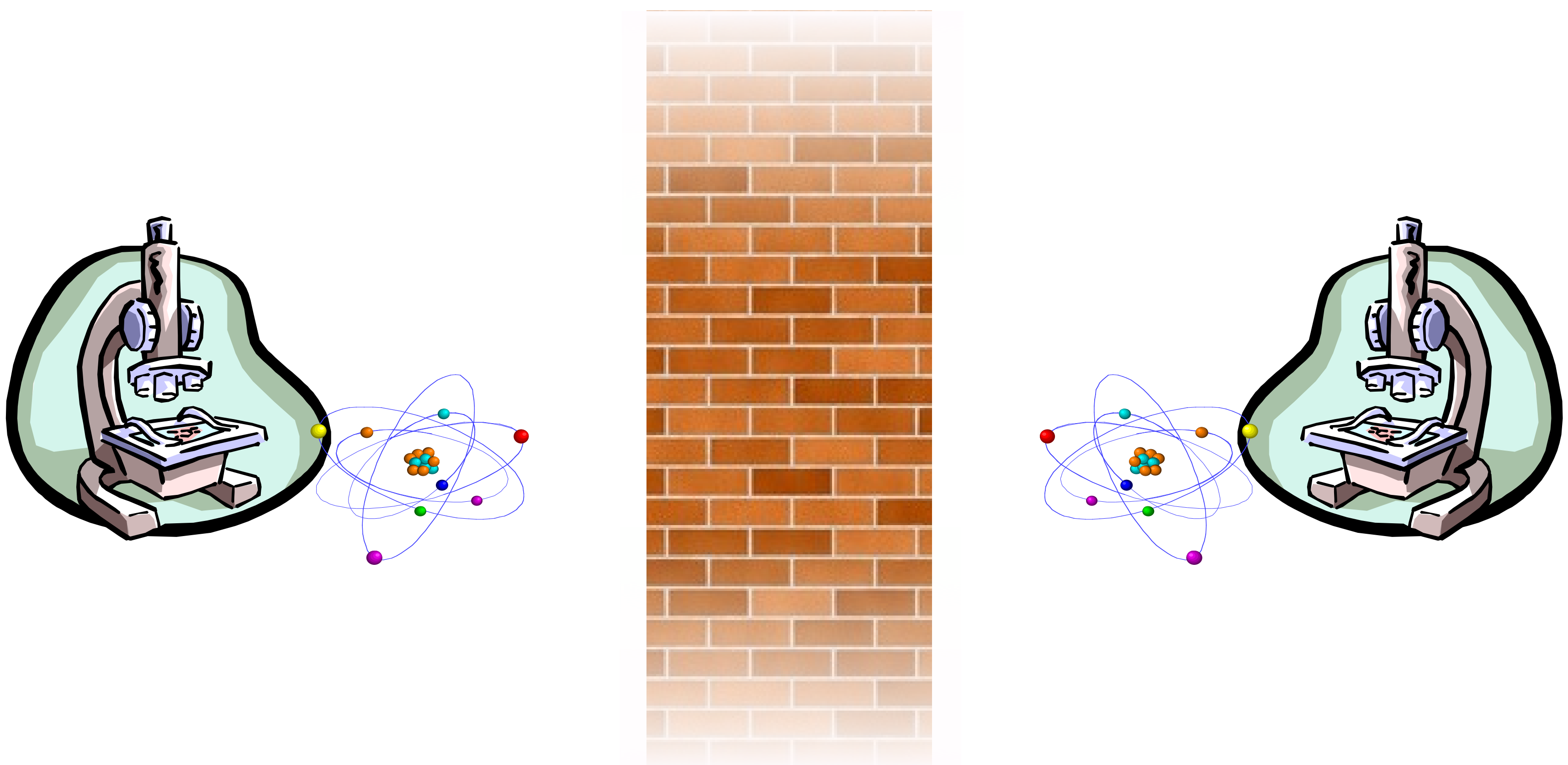
Quantifying entanglement with routine measurements: Of realists



Measure: $C_{zz}(\hat{\varrho}) = \mathrm{tr}\left[\hat{\varrho}\left(\hat{\sigma}_A^z \otimes \hat{\sigma}_B^z\right)\right] - \mathrm{tr}\left[\hat{\varrho}\hat{\sigma}_A^z\right]\mathrm{tr}\left[\hat{\varrho}\hat{\sigma}_B^z\right]$, ...

Observe: $C_{zz}(\hat{\varrho}) = -1$, ...

Know: $\mathrm{tr}\left[\hat{\varrho}^2\right] = 1$

$$|\psi\rangle = \frac{|01\rangle + \mathrm{e}^{\mathrm{i}\phi}|10\rangle}{\sqrt{2}}$$

Problem:

Experiment does not have direct access to entanglement, only to a limited set of measurement data.

General Question:

What is the *least* amount of entanglement compatible with measurement data?

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General Question:

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Measurement data, e.g., C_{zz}

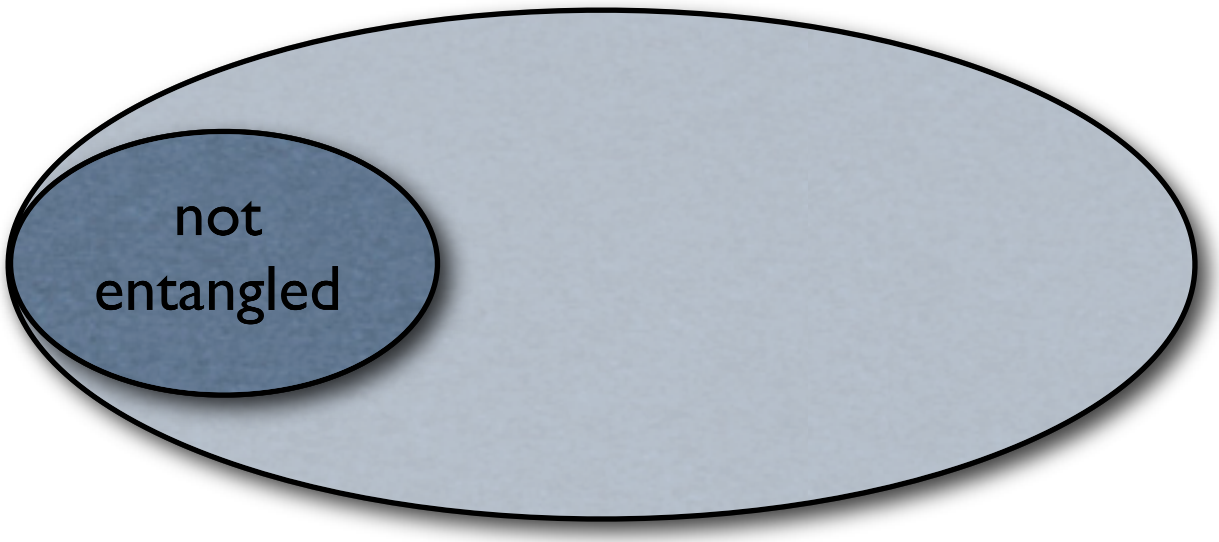
$$E_{\min} = \min_{\hat{\varrho}} \left\{ E(\hat{\varrho}) \; : \; \mathrm{tr}[\hat{\varrho}\hat{A}_i] = a_i \right\}$$

Entanglement measure

Result:

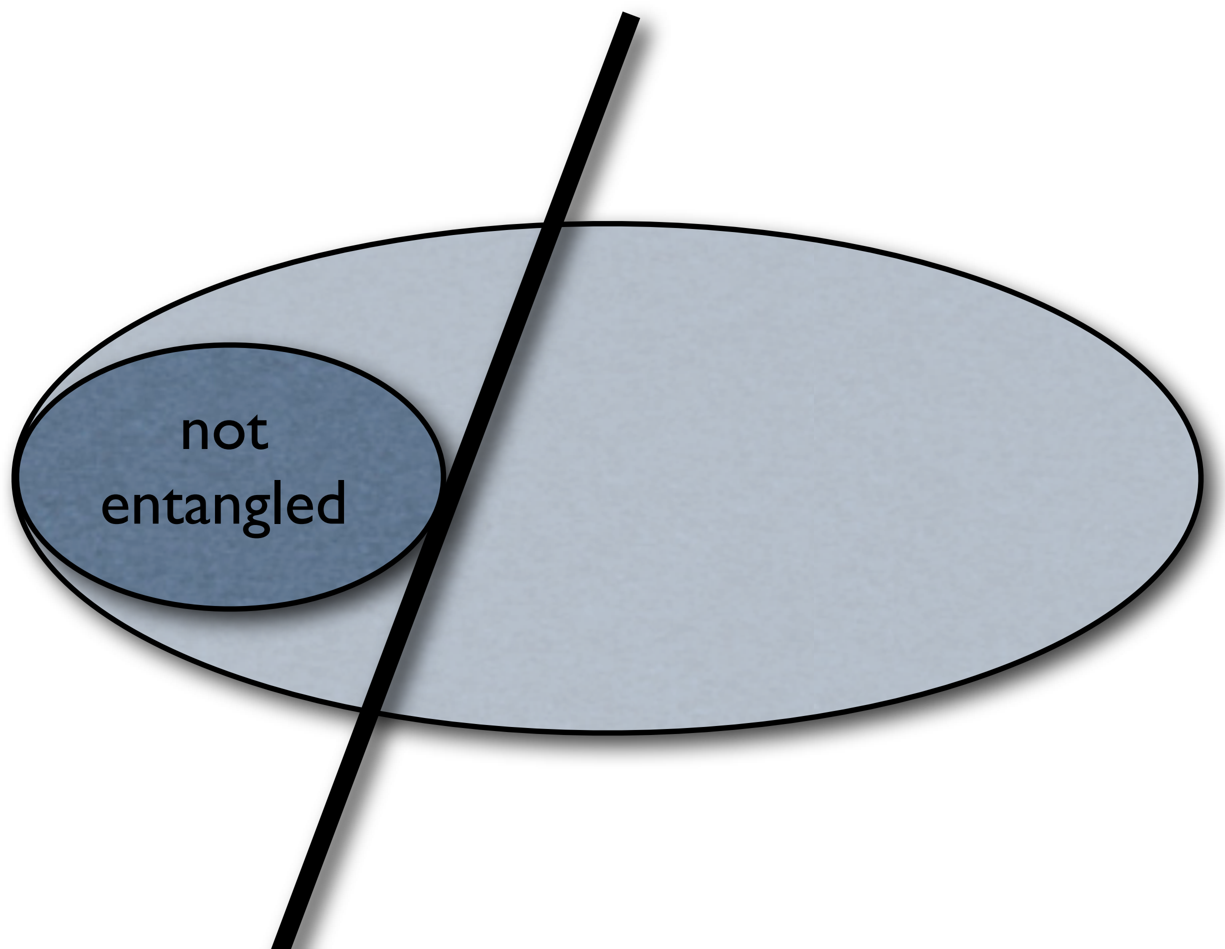
Measurement data guarantees that at least $E_{\min}$ units of entanglement are present.

Construct witness operator $\hat{W}$



Construct witness operator $\hat{W}$

$\mathrm{tr}[\hat{W}\hat{\varrho}] \geq 0$ for all separable states

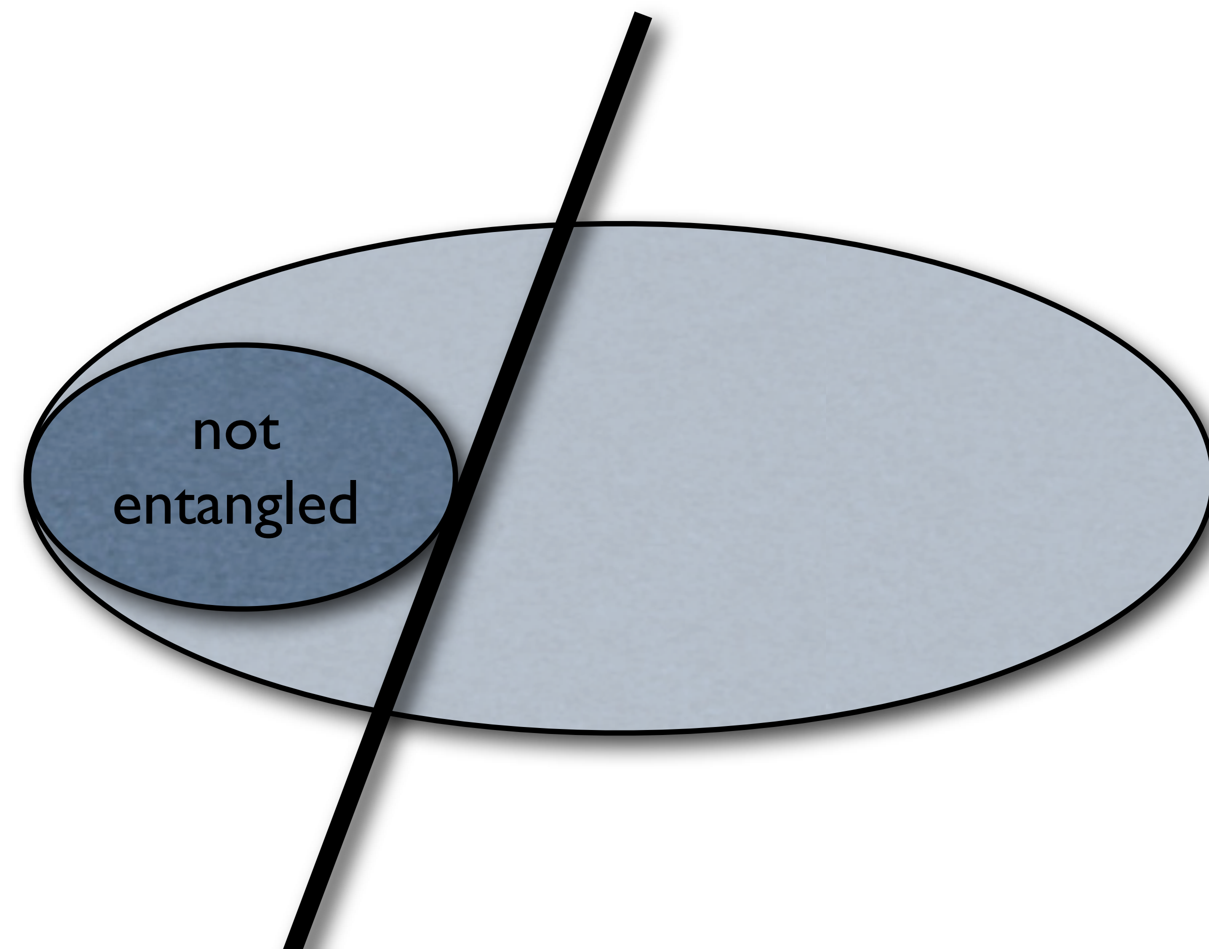


Quantifying entanglement with routine measurements: Witnesses

Construct witness operator $\hat{W}$

$\mathrm{tr}[\hat{W}\hat{\varrho}] \geq 0$ for all separable states

Decompose $\hat{W} = \sum_{i,j} w_{i,j} \hat{\sigma}_i \otimes \hat{\sigma}_j$

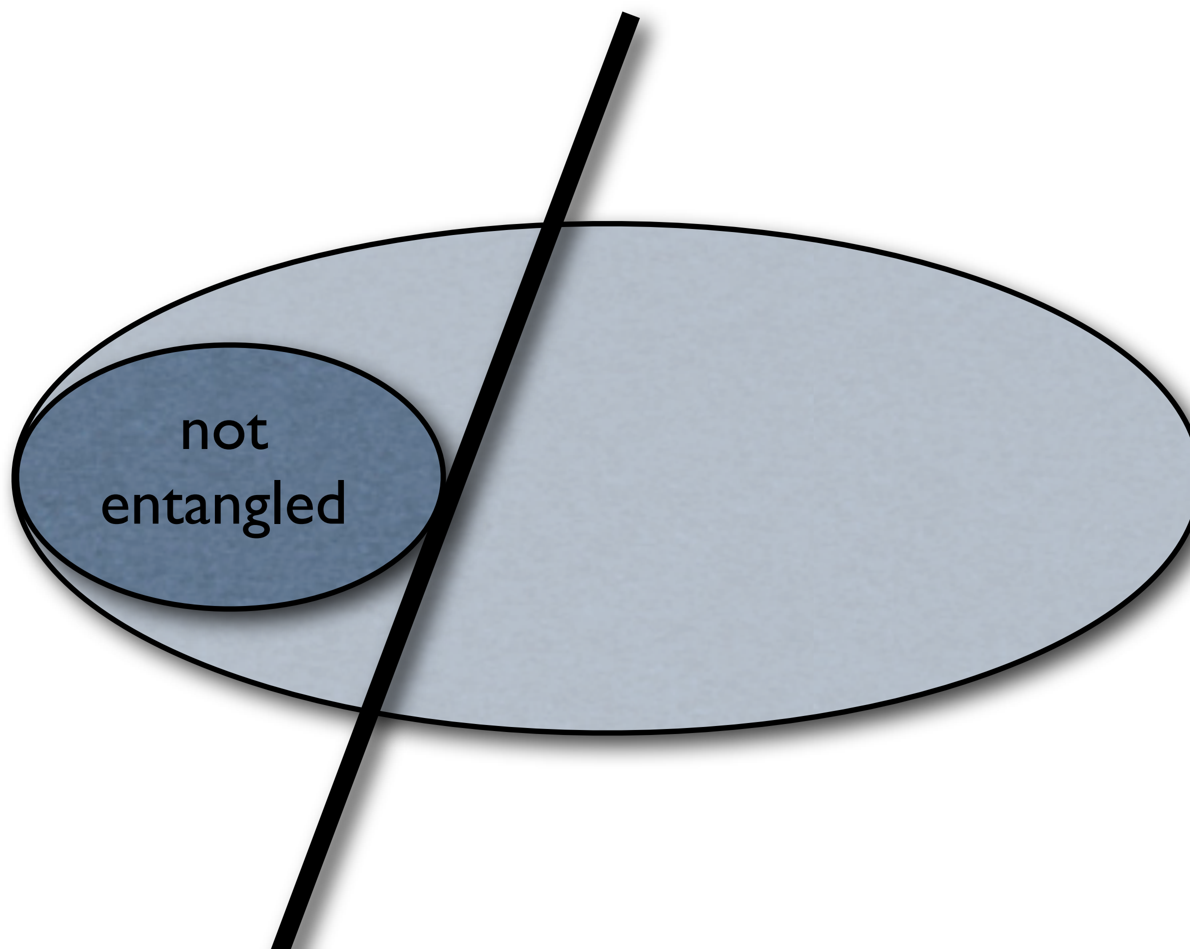


Construct witness operator $\hat{W}$

$\mathrm{tr}[\hat{W}\hat{\varrho}] \geq 0$ for all separable states

Decompose $\hat{W} = \sum_{i,j} w_{i,j} \hat{\sigma}_i \otimes \hat{\sigma}_j$

Measure $\mathrm{tr}[\hat{\sigma}_i \otimes \hat{\sigma}_j \, \hat{\varrho}]$



Construct witness operator $\hat{W}$

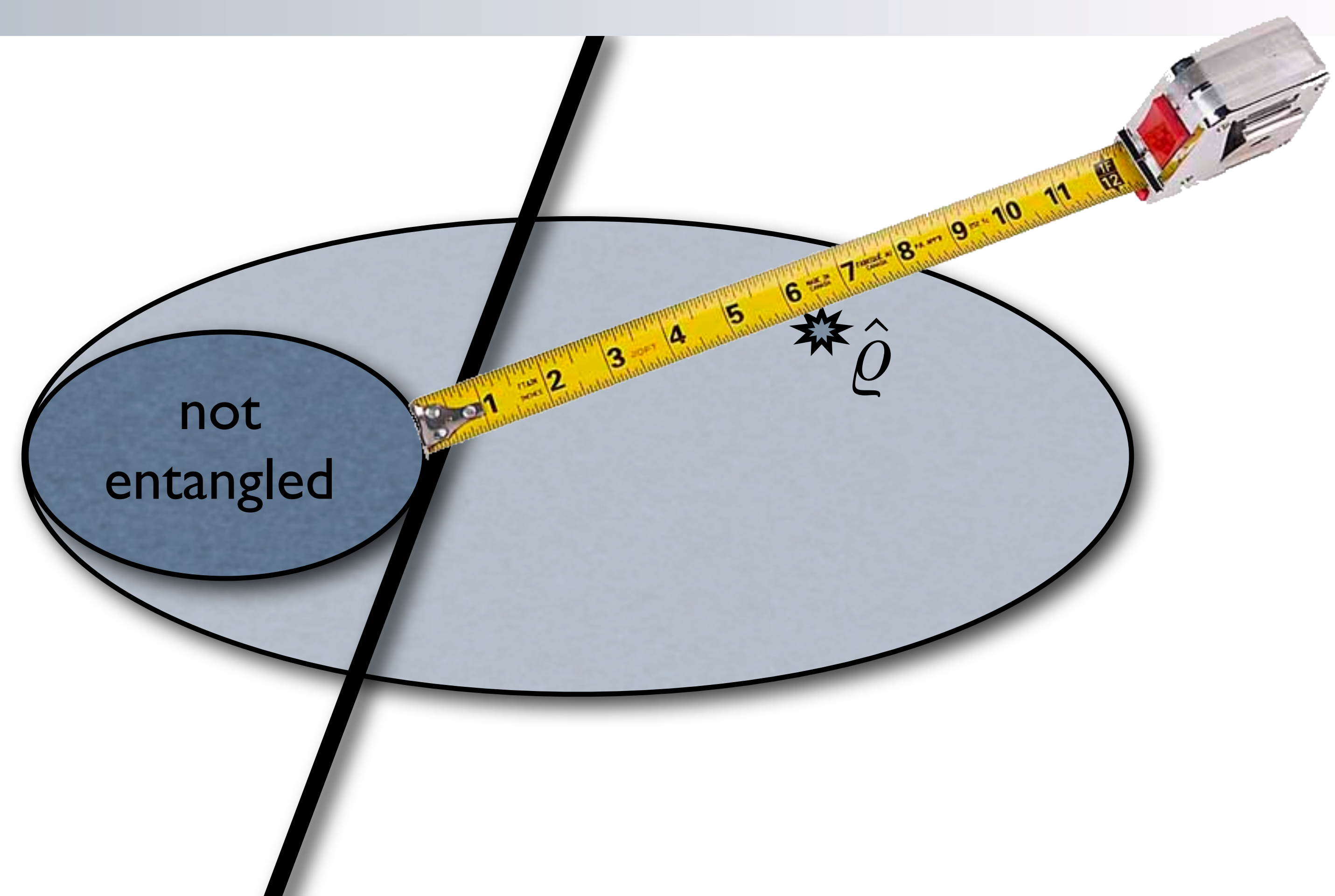
Decompose

- Only *yes/don't know* answer
- Why throw away information?

$$E_{\min} = \min_{\hat{\varrho}} \left\{ E(\hat{\varrho}) \;:\; \mathrm{tr}[\hat{\varrho}\hat{A}_i] = a_i \right\}$$

Measure $\mathrm{tr}[\hat{\sigma}_i \otimes \hat{\sigma}_j \, \hat{\varrho}]$

Recombine to $\mathrm{tr}[\hat{W}\,\hat{\varrho}]$



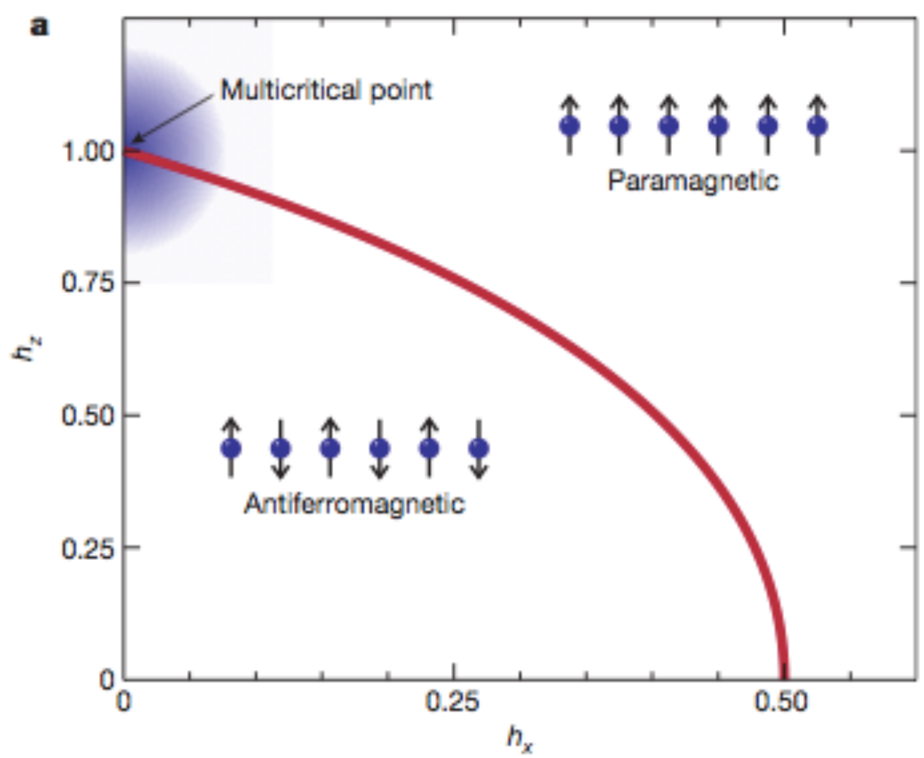
$E_{\min} = \min_{\hat{\varrho}} \left\{ E(\hat{\varrho}) \,:\, \mathrm{tr}[\hat{\varrho}\hat{A}_i] = a_i \right\}$ complicated optimization problem

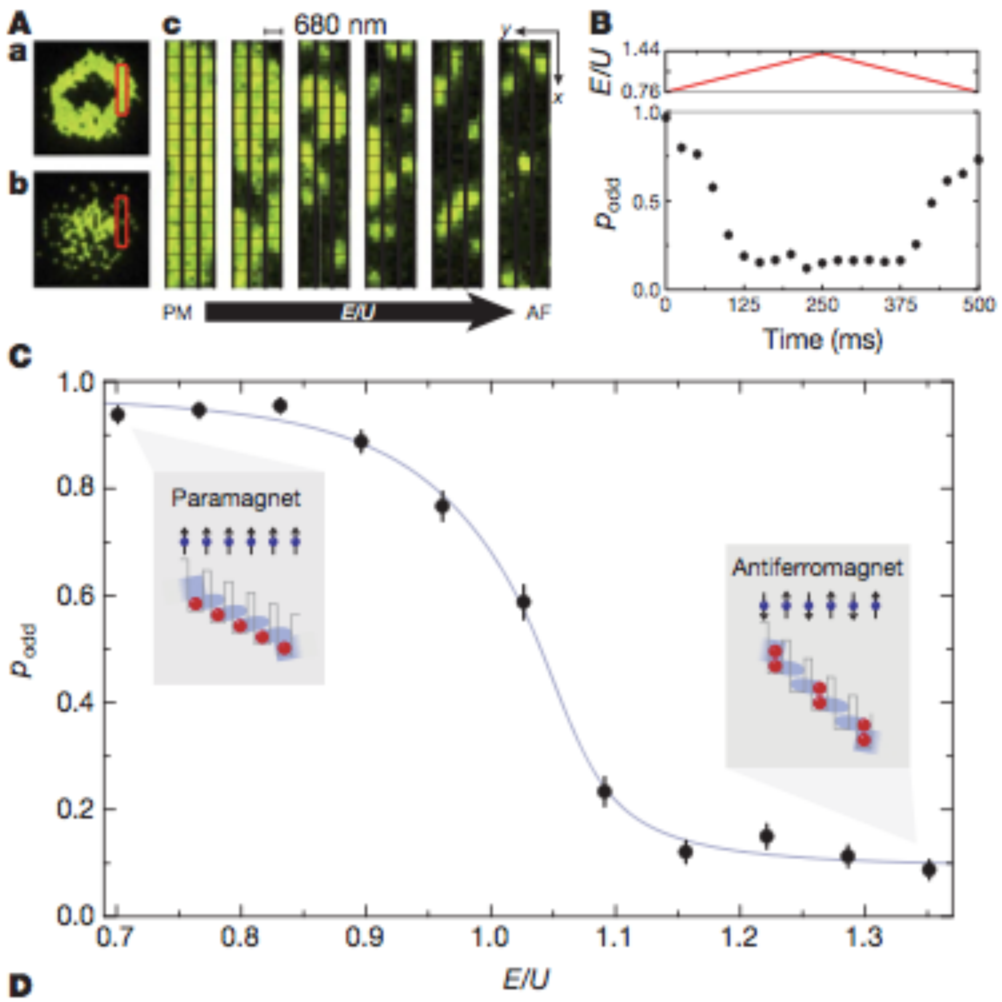
BUT may sometimes be formulated as a SDP:

$$E_{\min} \geq \log_2 \max \left\{ \nu_0 + \sum_{k=1}^{M} \nu_k a_k \,\middle|\, \nu_0 \mathbb{1} + \sum_{k=1}^{M} \nu_k \hat{A}_k \leq \hat{M}^{\Gamma},\ \hat{M}^{\dagger} = \hat{M},\ \|\hat{M}\|_{op} \leq 1,\ \nu_k \in \mathbb{R} \right\}$$

Audenaert & Plenio, NJP **8**, 266 (2006)

$$\hat{H} = \frac{1}{2}\sum_{i=1}^{N-1}\hat{\sigma}_i^z\hat{\sigma}_{i+1}^z - \sum_{i=1}^{N}\left(h_z\hat{\sigma}_i^z + h_x\hat{\sigma}_i^x\right)$$





Simon, Bakr, Ma, Tai, Preiss, Greiner, Nature **472**, 307 (2011)

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$$\sum_{i,j=1}^{N} \hat{\sigma}_i^x \hat{\sigma}_j^x \, , \quad \sum_{i,j=1}^{N} \hat{\sigma}_i^x \hat{\sigma}_j^y \, ,$$

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Quantifying entanglement with routine measurements: SDP and correlations

$$\hat{H} = \frac{1}{2}\sum_{i=1}^{N-1} \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z - \sum_{i=1}^{N} \left(h_z \hat{\sigma}_i^z + h_x \hat{\sigma}_i^x\right)$$

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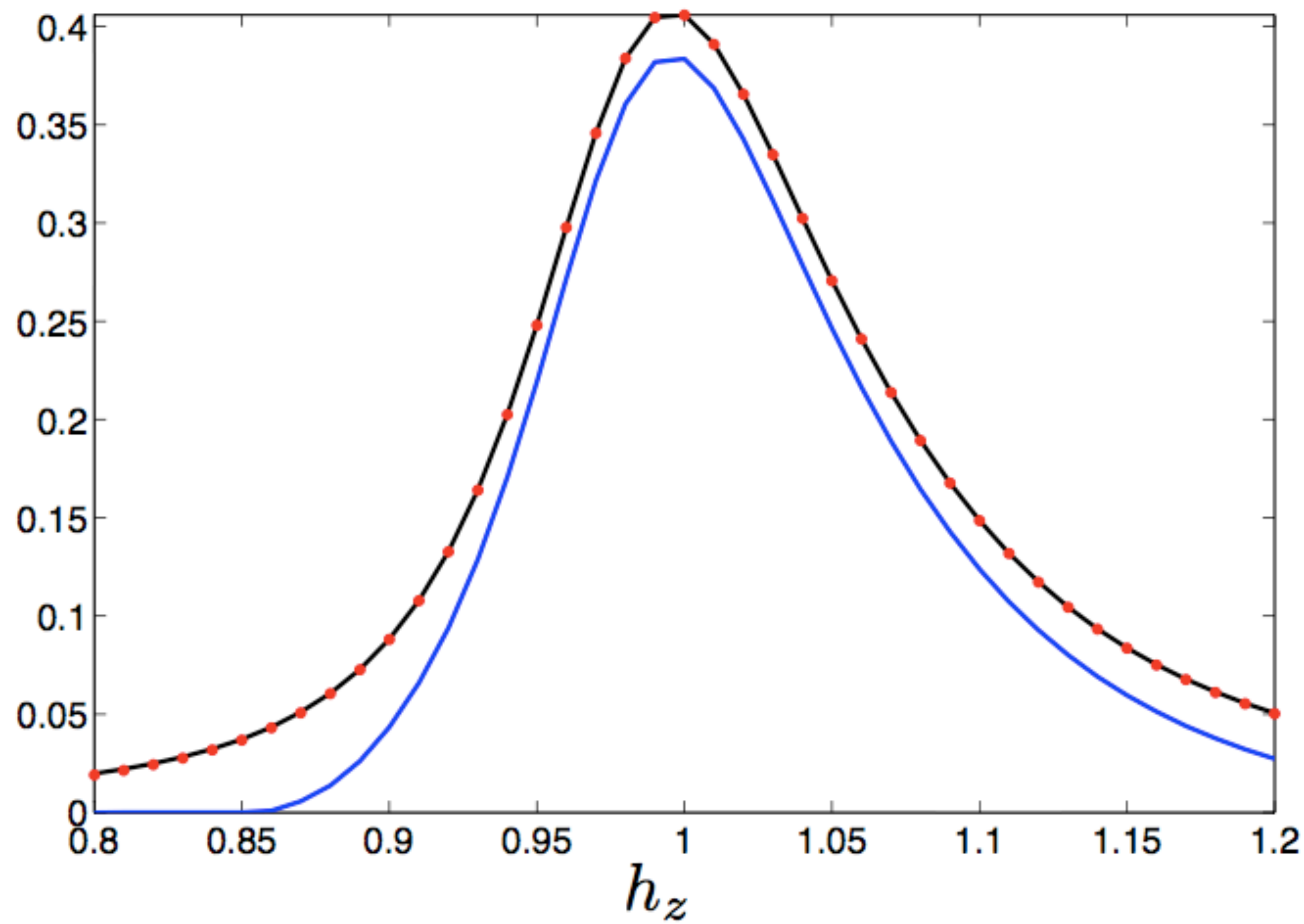

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or even be solved:

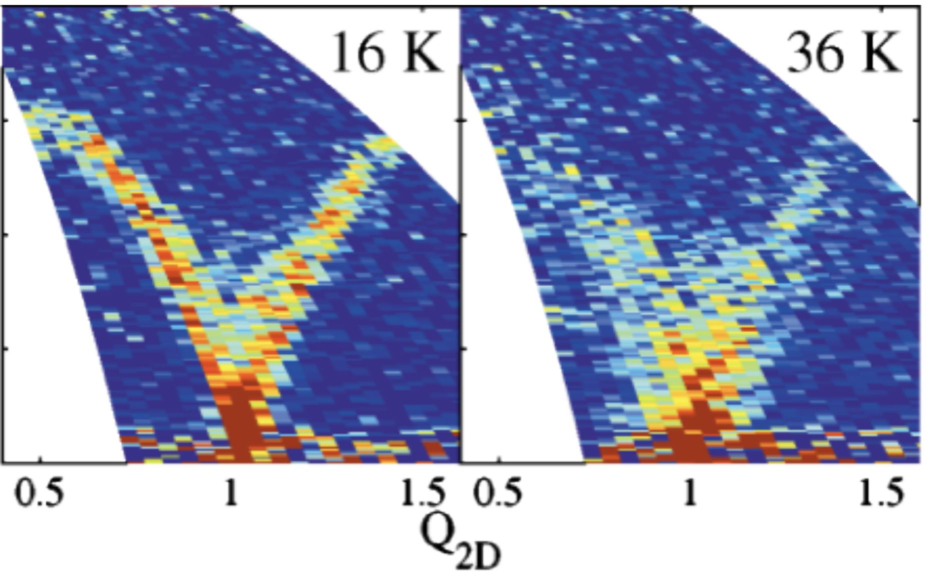
Minimal amount of entanglement compatible with

$C_{zz} = \mathrm{tr}[\hat{\varrho}\,(\hat{\sigma}_A^z \otimes \hat{\sigma}_B^z)]$ and $C_{xx} = \mathrm{tr}[\hat{\varrho}\,(\hat{\sigma}_A^x \otimes \hat{\sigma}_B^x)]$

is $E_{\min} = \max\left\{0, \log_2(|C_{xx}| + |C_{zz}|)\right\}$

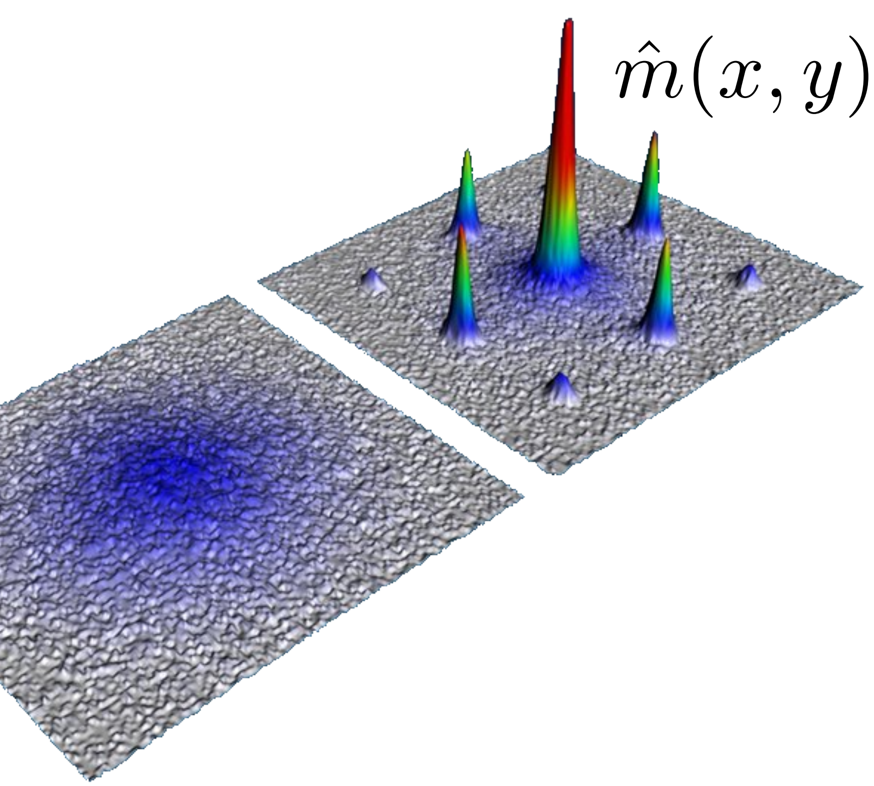
Audenaert & Plenio, NJP **8**, 266 (2006)

Quantifying entanglement with routine measurements: Structure factors



$$\hat{S}(q) = \sum_{i,j,\alpha} \mathrm{e}^{\mathrm{i}q(r_i - r_j)} \hat{\sigma}_i^\alpha \hat{\sigma}_j^\alpha$$

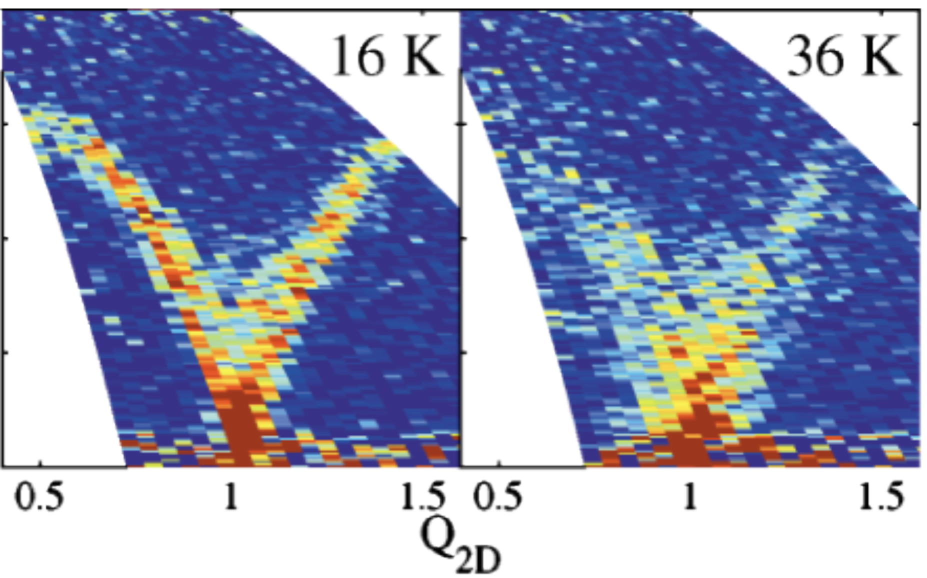
Cp: Krammer, Kampermann, Bruss, Bertlmann, Kwek, Macchiavello, PRL **103**, 100502 (2009)



$$\hat{m}(x,y) = \sum_{\substack{i_x, i_y \\ j_x, j_y}} f^*_{i_x,i_y}(x,y) f_{j_x,j_y}(x,y) \sum_{i_z} \hat{b}^\dagger_{(i_x\, i_y\, i_z)} \hat{b}_{(j_x\, j_y\, i_z)}$$

$$f_{i_x,i_y}(x,y) = f_{i_x}(x) f_{i_y}(y), \quad f_i(x) = \sum_{p \in \mathbb{Z}} \mathrm{e}^{2\pi \mathrm{i} p (x/a - i)/N} \mathrm{e}^{-4\mathrm{i} t E_R (p/N)^2/\hbar} c_p$$

Quantifying entanglement with routine measurements: Structure factors



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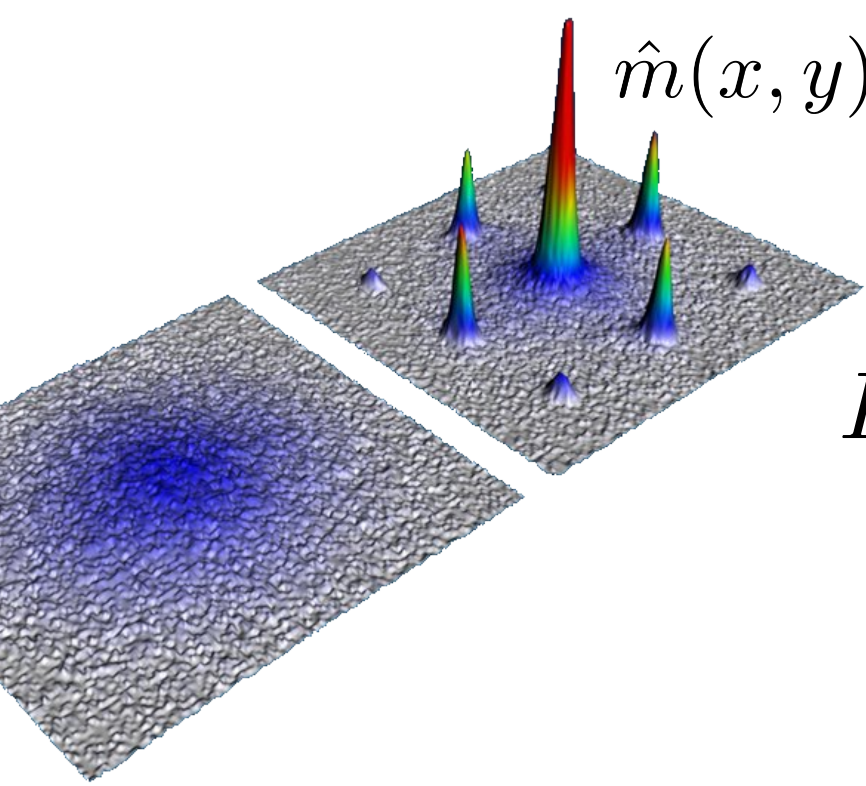

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Several entanglement measures may be expressed as

$$E_{\mathcal{C}}(\hat{\varrho}) = \max\left\{0, -\min_{\hat{W}\in\mathcal{W}\cap\mathcal{C}} \mathrm{tr}[\hat{W}\hat{\varrho}]\right\}, \quad \mathcal{W} : \langle\hat{W}\rangle_{\mathrm{sep}} \geq 0$$

Brandao, PRA **72** 022310 (2005)

Quantifying entanglement with routine measurements: Structure factors

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$\mathcal{C}:$ robustness of entanglement: $\langle\hat{W}\rangle_{sep} \leq 1$

generalized robustness: $\mathbb{1} - \hat{W} \geq 0$

best separable approximation: $\mathbb{1} + \hat{W} \geq 0$

family of monotones: $-a\mathbb{1} \leq \hat{W} \leq b\mathbb{1}$

Brandao, PRA **72** 022310 (2005)

$$E_{\min} = \min_{\hat{\varrho}} \left\{ E(\hat{\varrho}) \ : \ \mathrm{tr}\left[\hat{\varrho}\hat{S}(q)\right] = S(q) \right\}$$
$$\geq \max\left\{0, 1 - \frac{S(q)}{2N}\right\}$$

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Quantifying entanglement with routine measurements: Structure factors

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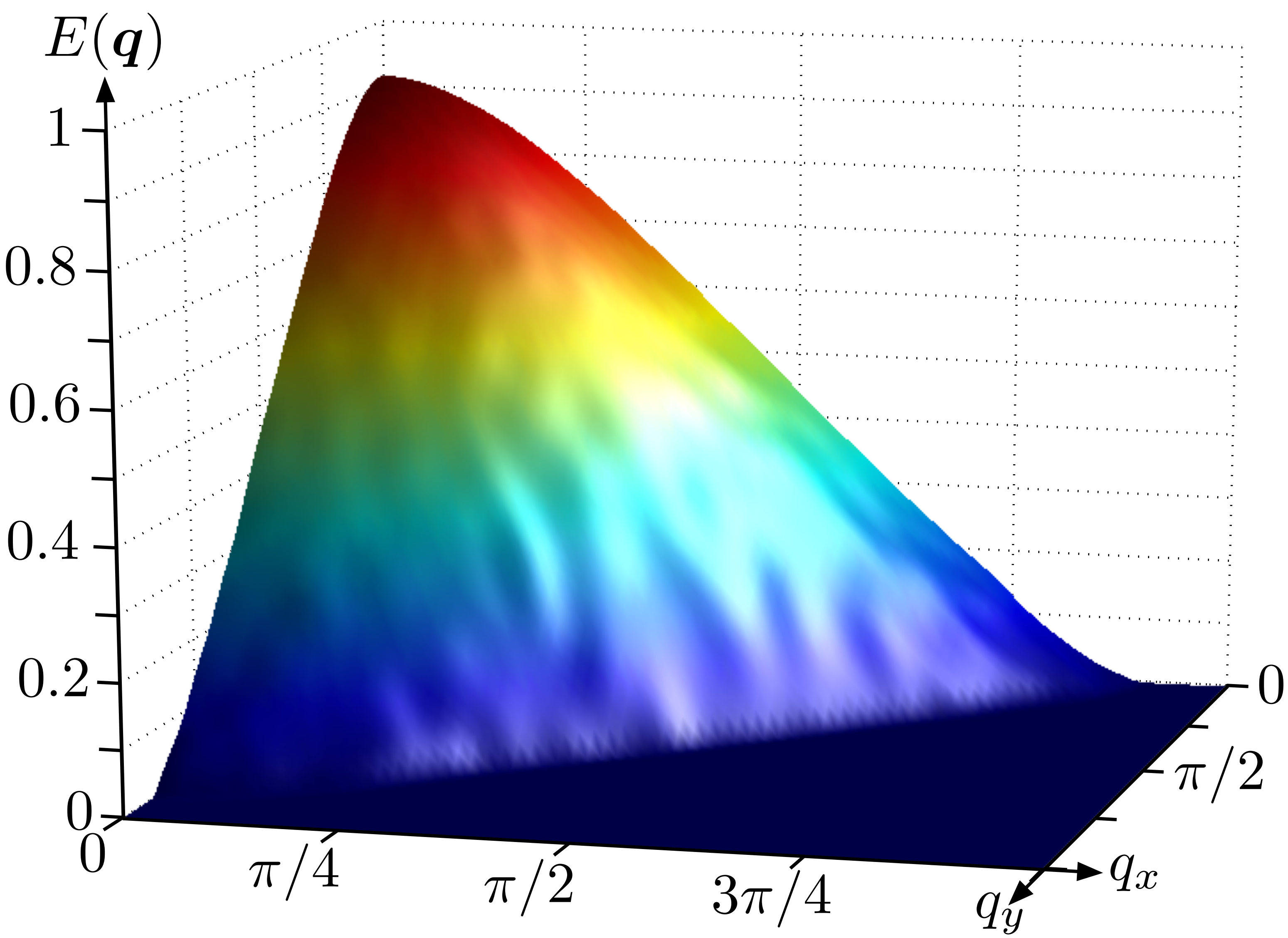

$$\hat{\varrho} = \frac{\mathrm{e}^{-\hat{H}/(k_B T)}}{Z}$$

$$\hat{H} = J \sum_{\alpha} \sum_{\langle i,j \rangle} \hat{\sigma}_i^{\alpha} \hat{\sigma}_j^{\alpha}$$

$$N = 30 \times 30$$

$$\frac{J}{k_B T} = 1$$

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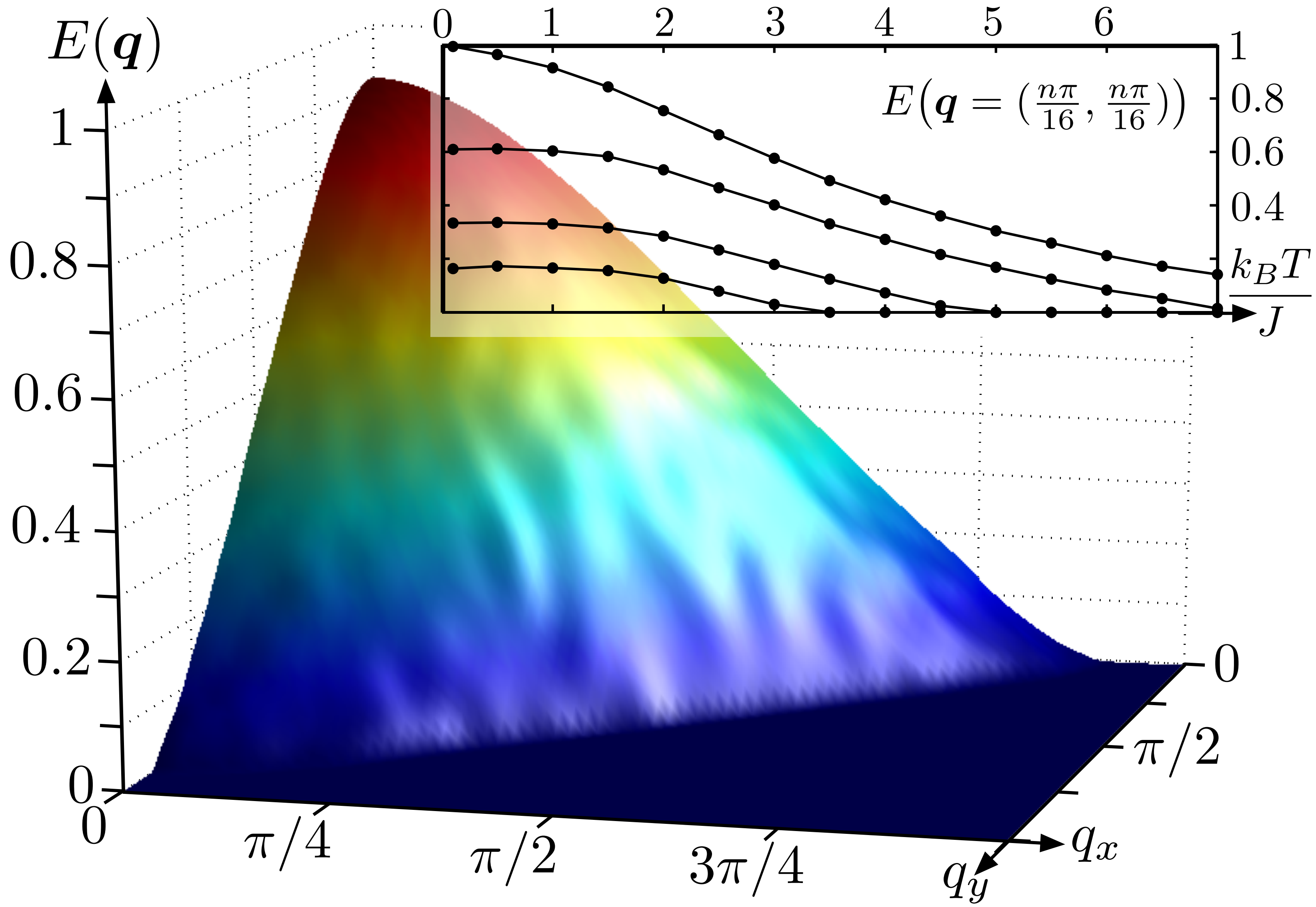

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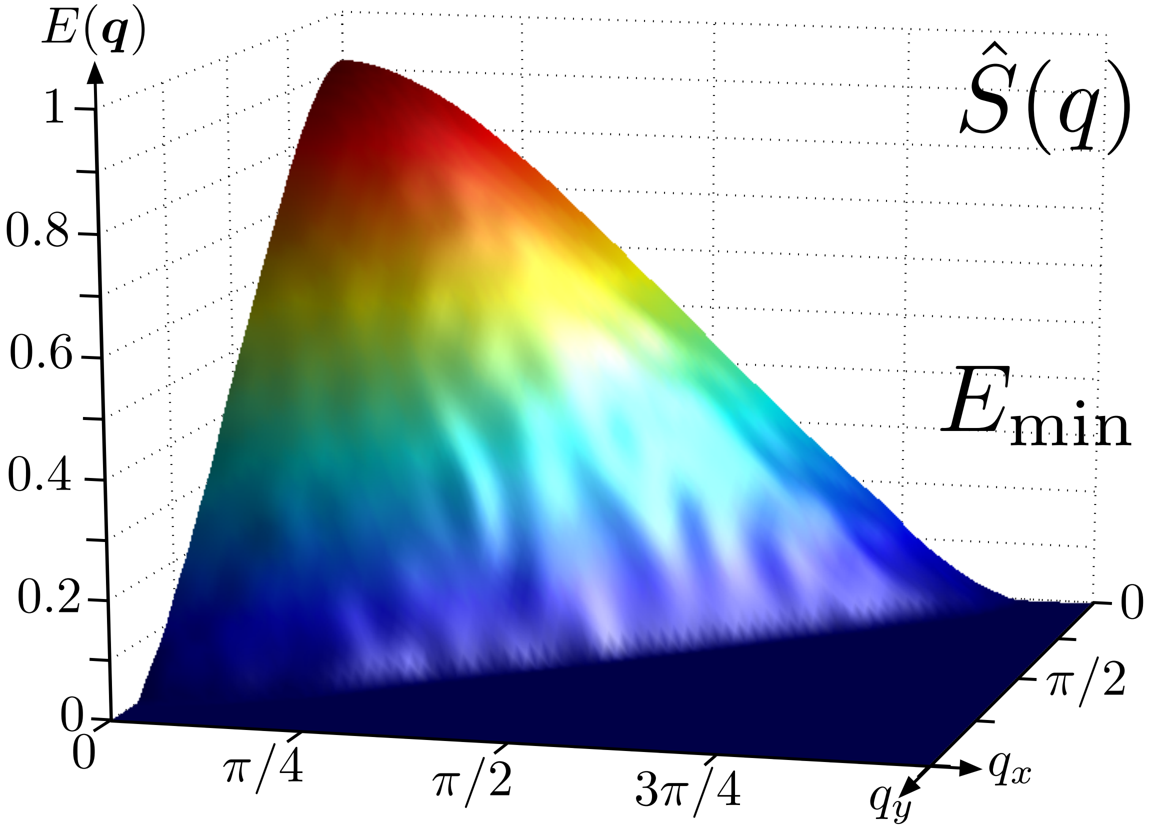
Quantifying entanglement with routine measurements: Structure factors



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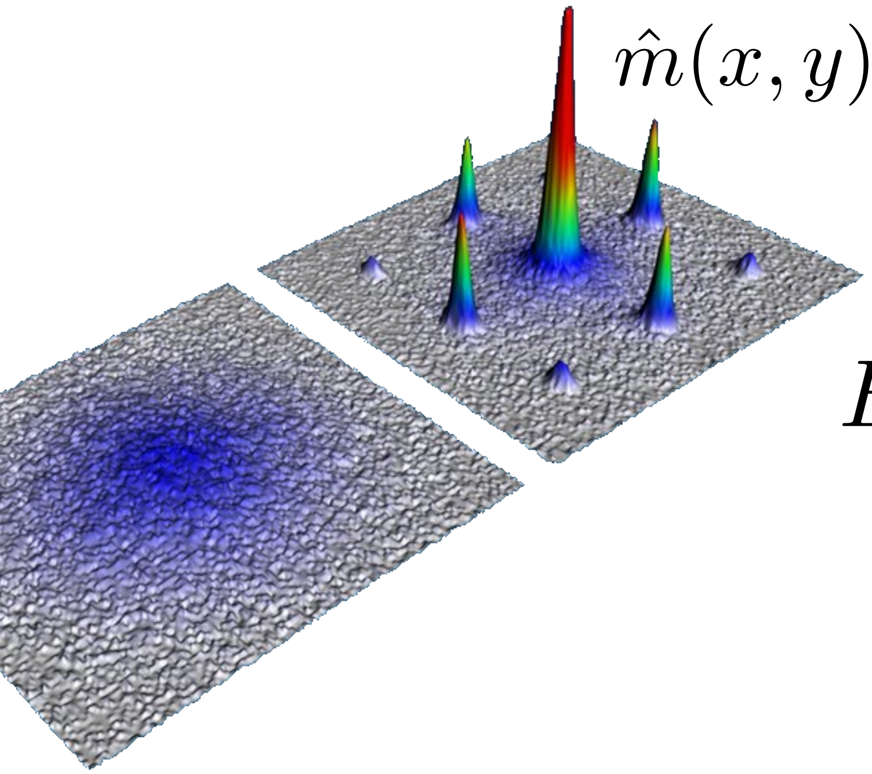

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We construct SSR entanglement monotone:

$$E(\hat{\varrho}) = \max\left\{ 0, -\sum_{N=0}^{\infty} \min_{\hat{W}\in\mathcal{C}_N} \mathrm{tr}[\hat{P}_N \hat{\varrho} \hat{P}_N \hat{W}] \right\}$$

$\mathcal{C}_N$: $cN \pm \hat{W} \geq 0$ and $\mathrm{tr}[\hat{\sigma}_N \hat{W}] \geq 0$ for $\hat{\sigma}_N$ separable, locally respecting SSR

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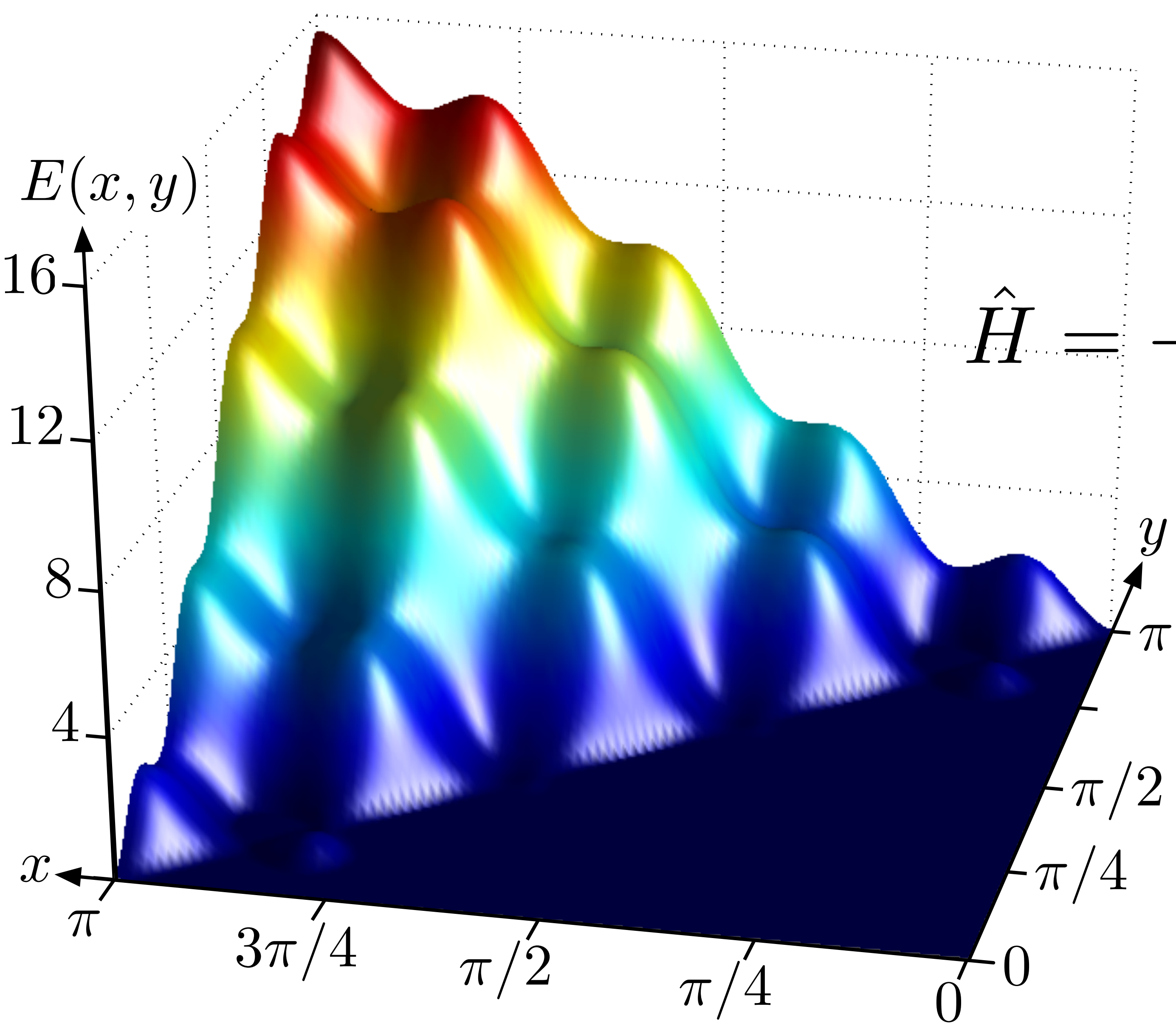

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$$N = 10 \times 10 \times 10$$

$$\langle \hat{n}_i \rangle = 1$$

$$\frac{U}{k_B T} = \frac{1}{5}$$

$$\frac{J}{U} = 0.01$$

Quantifying entanglement with routine measurements: Structure factors

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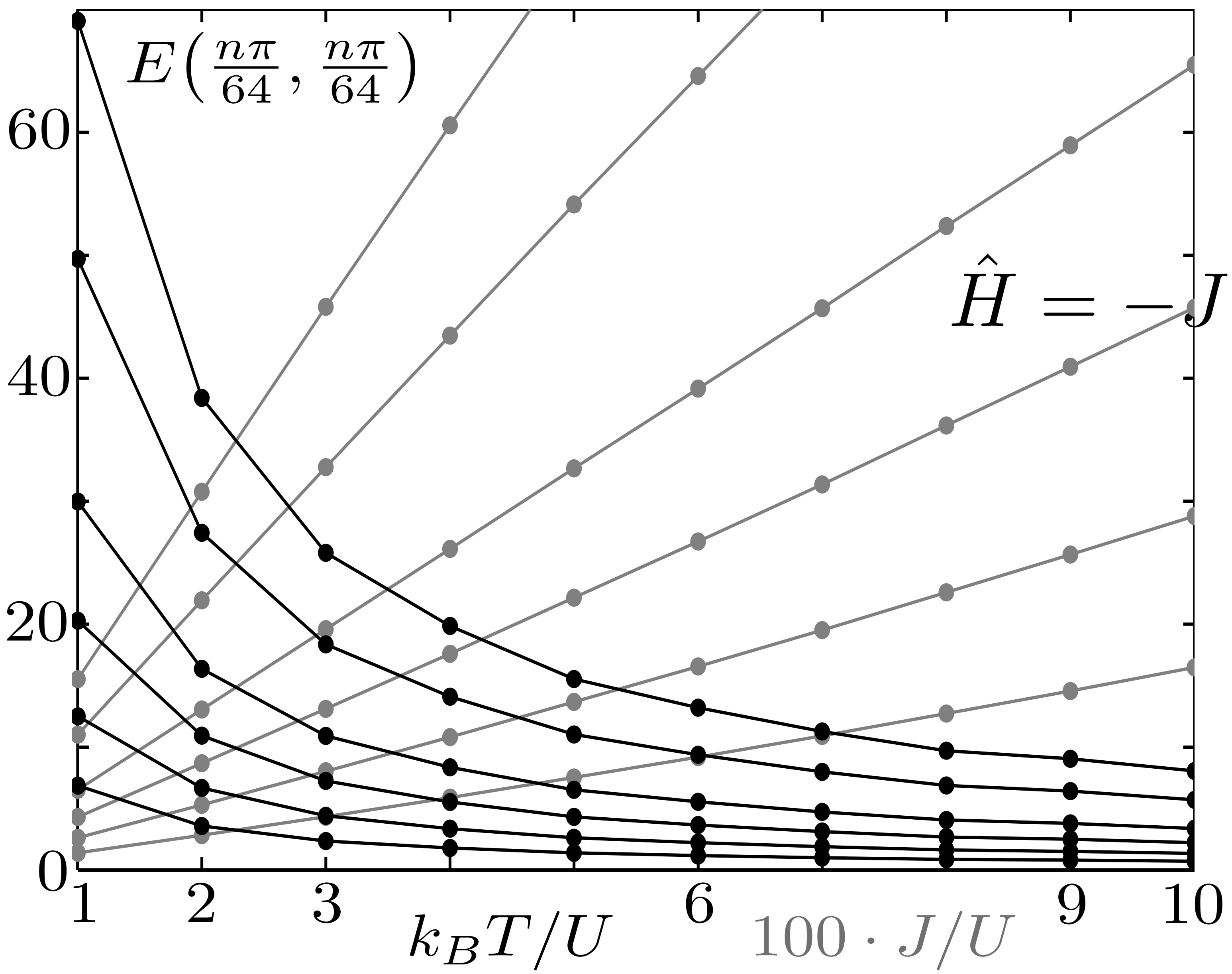

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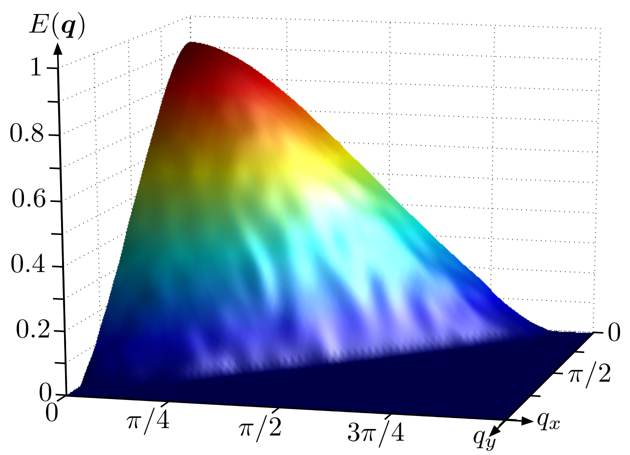

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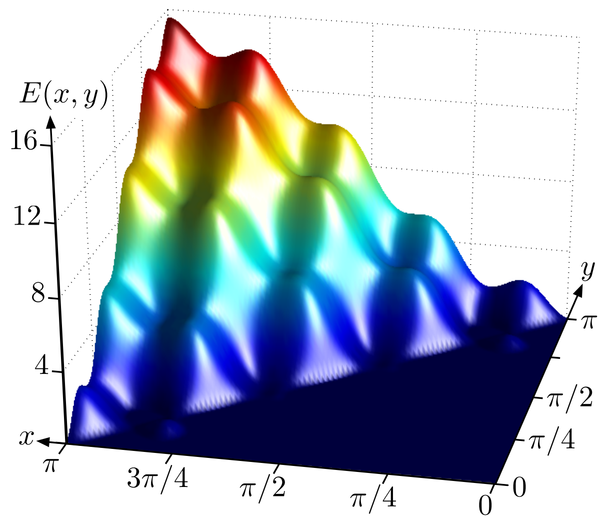

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Quantification of entanglement in many-body systems

- without assumptions
- using standard measurements only
- versatile enough to accommodate different measurements