Lecture 1

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Quantum Enhanced Metrology

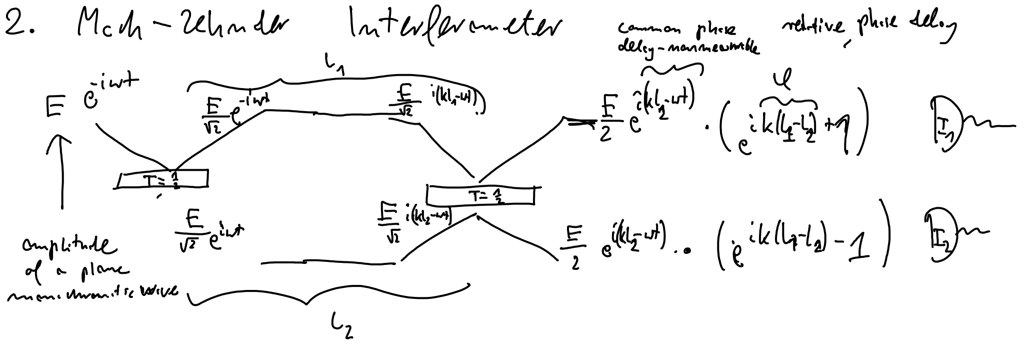
1. Introduction

Metrology – science of measurement
in particular: designing measurement schemes reaching best possible precision.

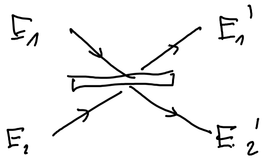
Quantum enhanced metrology – achieve the precision restricted only by the laws of quantum mechanics.

One of the most important tools for high precision measurements of e.g. length is interferometry.

2. Mach–Zehnder Interferometer



Action of an ideal (lossless) beamsplitter



$$\begin{bmatrix} E_1' \\ E_2' \end{bmatrix} = \begin{bmatrix} r & t \\ t & -r \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}$$

$$R = r^2 \quad T = t^2 \quad R+T=1$$

more generally $\begin{bmatrix} re^{i\theta_1} & te^{i\theta_2} \\ te^{-i\theta_2} & -re^{-i\theta_1} \end{bmatrix}$

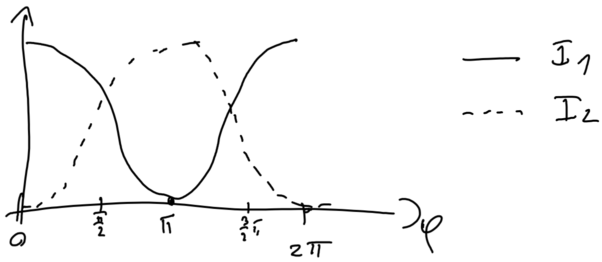
$\varphi = k\cdot\Delta L = \frac{2\pi}{\lambda}\Delta L$

$$I_1 = \frac{1}{4}|E|^2\cdot|e^{i\varphi}+1|^2 = \frac{1}{2}|E|^2(1+\cos\varphi) = |E|^2\cos^2\frac{\varphi}{2}$$

$$I_2 = \frac{1}{4}|E|^2\,|e^{i\varphi}-1|^2 = \frac{1}{2}|E|^2(1-\cos\varphi) = |E|^2\sin^2\frac{\varphi}{2}$$

Measuring I_1, I_2 we can estimate φ, and assuming we know λ we can learn ΔL.
What limits the precision of estimating φ?

3. Estimating φ using classical light



There is an ambiguity φ, $2\pi-\varphi$ give the same I_1, I_2
but this is not a problem it is enough to measure
e.g. two times introducing additional known phase shift.

Apart from that if we knew I_1, I_2 perfectly, we would learn φ perfectly $\cos\varphi = \frac{I_1 - I_2}{|E|^2}$ $\varphi = \arccos\frac{I_1 - I_2}{|E|^2}$

But I_1, I_2 are never known perfectly ...

Light consist of photons, intensity is proportional to the number of photons absorbed $I \sim n$

But n is discrete so we will not get arbitrary good precision.
And what is more important: classical states of light have Poissonian statistics of photon number distribution.

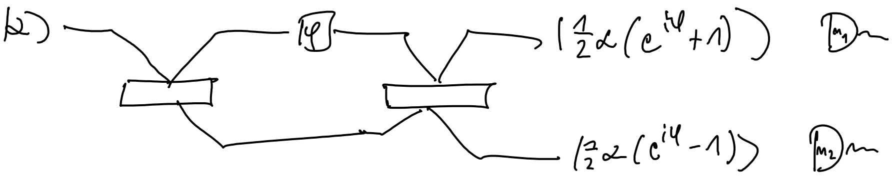
$|\alpha\rangle$ - coherent state representing classical state of light. α - amplitude normalized such that $|\alpha|^2$ - mean number of photons

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle \qquad p_n = |\langle n|\alpha\rangle|^2 = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!}$$

Poissonian statistics

$$\langle n\rangle = |\alpha|^2, \quad \Delta^2 n = \langle n^2\rangle - \langle n\rangle^2 = \langle n\rangle = |\alpha|^2$$

shot noise



One registers n_1, n_2 which are governed by Poissonian distributions with $\langle n_1\rangle = \frac{1}{2}|\alpha|^2(1+\cos\varphi)$ $\langle n_2\rangle = \frac{1}{2}|\alpha|^2(1-\cos\varphi)$

$$p_{n_1 n_2} = e^{-\langle n_1\rangle} \frac{\langle n_1\rangle^{n_1}}{n_1!} \cdot e^{-\langle n_2\rangle} \frac{\langle n_2\rangle^{n_2}}{n_2!}$$

If we infer $\cos\tilde\varphi := \frac{n_1 - n_2}{|\alpha|^2}$, φ will fluctuate due to n_1, n_2 fluctuations.
What is the estimation uncertainty?
Calculate the variance of $\cos\varphi$:

$$\Delta^2 \cos\tilde\varphi = \langle \cos^2\tilde\varphi\rangle - \langle\cos\tilde\varphi\rangle^2 = \frac{1}{|\alpha|^4} \cdot \left(\langle n_1^2\rangle + \langle n_2^2\rangle - 2\langle n_1 n_2\rangle\right) -$$

$$- \frac{1}{|\alpha|^4}(\langle n_1\rangle - \langle n_2\rangle)^2 = \frac{1}{|\alpha|^4}\left(\Delta^2 n_1 + \Delta^2 n_2\right) = \frac{1}{|\alpha|^4}(\langle n_1\rangle + \langle n_2\rangle)$$

$$-\frac{1}{|\alpha|^4}\cdot\left(\langle m_1\rangle-\langle m_2\rangle\right)^2 = \frac{1}{|\alpha|^4}\cdot\left(\Delta^2 m_1+\Delta^2 m_2\right) = \frac{1}{|\alpha|^4}\left(\langle m_1\rangle+\langle m_2\rangle\right)$$

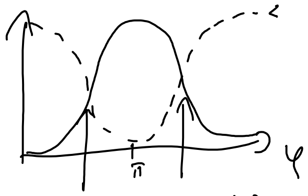
{ since m_1, m_2 independent

$$= 2\frac{1}{|\alpha|^4}\left[(1+\cos\varphi)+(1-\cos\varphi)\right] = \frac{1}{|\alpha|^2} = \frac{1}{\langle n\rangle}$$

$\langle n\rangle$ mean number of photons used.

$$\underbrace{\Delta^2\varphi = \frac{\Delta^2\cos\varphi}{\left(\frac{\partial\cos\varphi}{\partial\varphi}\right)^2} = \frac{1}{\langle n\rangle\sin^2\varphi}}$$

- $\frac{1}{\langle n\rangle}$ precision scaling (shot noise scaling)
- precision depends on the true value of φ

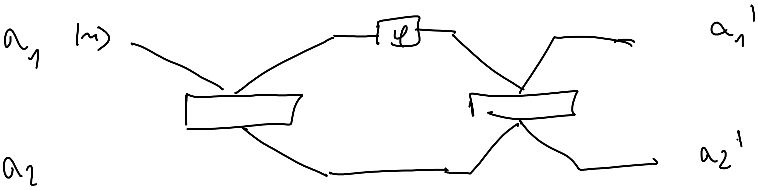


optimal precision curves are the steepest.

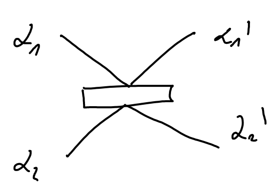
Estimation around π, 2π seems impossible. – – fluctuations are φ independent while mean photon numbers do not locally change here

Can we improve the precision by using quantum states of light

4. Estimation of φ using Fock states.



How do we describe evolution of quantum states



We know how classical amplitudes evolve

$$\alpha_i' = V_{ij}\alpha_j$$

$$|\alpha_i'\rangle = |\mathcal{U}_{ij}\alpha_j\rangle \qquad \} \text{ dicitamy } a_i$$

$$\alpha_i = \langle\psi|a_i|\psi\rangle \qquad \alpha_i' = \langle\psi|\mathcal{U}^\dagger a_i\mathcal{U}|\psi\rangle = V_{ij}\alpha_j$$

this implies that $\mathcal{U}^\dagger a_i\mathcal{U} = V_{ij}a_j = a_i'$ W

obracie Heisenberga op. anihilacji ewoluja tak jak klasyczne amplitudy

In terms of creation operators: $\left\{ V_{ij}^* a_j^\dagger = a_i'^\dagger \right.$, $V_{ki}^{*-1}V_{ij}^* a_j^\dagger = (V_{ki}^{*-1})a_i'^\dagger$

$$a_k^\dagger = V_{ik}a_i'^\dagger$$

To see how a given state of light evolves we just express input operators using the output ones

In Mach-Zehnder:

$$V = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} e^{i\varphi} & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} =$$

$$= \frac{1}{2} \cdot \begin{bmatrix} e^{i\varphi} & 1 \\ e^{i\varphi} & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \cdot \begin{bmatrix} e^{i\varphi}+1 & e^{i\varphi}-1 \\ e^{i\varphi}-1 & e^{i\varphi}+1 \end{bmatrix} = e^{\frac{i\varphi}{2}} \cdot \begin{bmatrix} \cos\frac{\varphi}{2} & i\sin\frac{\varphi}{2} \\ i\sin\frac{\varphi}{2} & \cos\frac{\varphi}{2} \end{bmatrix}$$

So:

$$a_1^{+} = V_{11} a_1'^{+} + V_{21} a_2'^{+} = \underbrace{\tfrac{1}{2}\left(e^{i\varphi}+1\right)}_{c_1} a_1'^{+} + \underbrace{\tfrac{1}{2}\left(e^{i\varphi}-1\right)}_{c_2} a_2'^{+} =$$

$$= e^{\frac{i\varphi}{2}} \cdot \left(\cos\tfrac{\varphi}{2}\, a_1'^{+} + i\sin\tfrac{\varphi}{2}\, a_2'^{+} \right)$$

$$|n\rangle_{in} = \frac{a_1^{+n}}{\sqrt{n!}} |vac\rangle = \frac{\left(c_1 a_1'^{+} + c_2 a_2'^{+}\right)^n}{\sqrt{n!}} |vac\rangle =$$

$$= \sum_{k=0}^{n} \frac{1}{\sqrt{n!}} \binom{n}{k} c_1^k c_2^{n-k} \sqrt{k!}\sqrt{(n-k)!}\, |k, n-k\rangle_{out} =$$

$$= e^{\frac{in\varphi}{2}} \sum_{k=0}^{n} \sqrt{\binom{n}{k}} (\cos\varphi)^k (\sin\varphi)^{n-k} (i)^{n-k} |k, n-k\rangle$$

$$p_{n_1, n_2} = \delta_{n_1+n_2, n} \cdot \binom{n}{n_1} \left(\cos^2\tfrac{\varphi}{2}\right)^{n_1} \left(\sin^2\tfrac{\varphi}{2}\right)^{n_2}$$

$$\langle n_1\rangle = \sum_{n_1=0}^{\infty} n_1 \frac{n!}{n_1!(n-n_1)!} \cos^2\varphi^{n_1} \sin^2\varphi^{n-n_1} = n\cos^2\tfrac{\varphi}{2}$$

$$\langle n_2\rangle = \qquad = n\sin^2\tfrac{\varphi}{2}$$

We use the same estimator : $\widetilde{\cos\varphi} = \frac{n_1 - n_2}{n}$

$$\Delta^2(\widetilde{\cos\varphi}) = \frac{1}{n^2} \cdot \left(\langle (n_1 - n_2)^2\rangle - \left(\langle n_1\rangle - \langle n_2\rangle\right)^2 \right) =$$

$$= \frac{1}{n^2} \cdot \left(\sum_{n_1} \cdot (2n_1 - n)^2 \binom{n}{n_1} \left(\cos^2\tfrac{\varphi}{2}\right)^{n_1} \left(\sin^2\tfrac{\varphi}{2}\right)^{n-n_1} - n^2\cos^2\varphi \right) =$$

$$= \frac{1}{n^2} \cdot \left(n^2 - 4n^2\cos^2\tfrac{\varphi}{2} + 4n\cos^2\tfrac{\varphi}{2} \cdot \left(1 + (n-1)\cos^2\tfrac{\varphi}{2}\right) - n^2\cos^2\varphi \right) =$$

$$\langle n_1^2\rangle = \sum_{n_1} n_1^2 \frac{n!}{n_1!(n-n_1)!} \cos^2\tfrac{\varphi}{2}^{n_1} \sin^2\tfrac{\varphi}{2}^{n-n_1} = \cos^2\tfrac{\varphi}{2} \sum_{n_1} n_1 \cdot n \frac{(n-1)!}{(n_1-1)!(n-1-n_1+1)!} \cos^2\tfrac{\varphi}{2}^{n_1-1} \sin^2\tfrac{\varphi}{2}^{n-1-n_1+1}$$

$$= \cos^2\tfrac{\varphi}{2}\, n \cdot \sum_{n_1} \left[(n_1 - 1) + 1\right] \frac{(n-1)!}{(n_1-1)!(n-1-n_1+1)!} \cos^2\tfrac{\varphi}{2}^{n_1-1} \sin^2\tfrac{\varphi}{2}^{n-1-n_1+1} =$$

$$= \cos^2\tfrac{\varphi}{2} \cdot n \cdot \left(1 + (n-1) \cdot \cos^2\tfrac{\varphi}{2} \right)$$

$$= \frac{1}{n^2} \cdot \left(n^2 - 4n^2\cos^2\tfrac{\varphi}{2} + 4n\cos^2\tfrac{\varphi}{2} + 4n^2\cos^4\tfrac{\varphi}{2} - 4n\cos^4\tfrac{\varphi}{2} - n^2\left(2\cos^2\tfrac{\varphi}{2} - 1\right)^2 \right)$$

$$= \frac{1}{n^2} \cdot \left(n\sin^2\varphi \right) = \frac{\sin^2\varphi}{n}$$ fluktuacje zależne od φ

So:

$$\Delta^2\varphi = \frac{\Delta^2\cos\varphi}{\left(\frac{d\cos\varphi}{d\varphi}\right)^2} = \frac{1}{n}$$ nie zależy od φ

The same scaling as for coherent state but now there is no dependence on φ.

Notice that a coherent state behaves in the same way as incoherent mixture of Fock states ...

states with Poissonian statistics. It should not be surprising that mixing introduces some additional difficulties

5. Estimation using squeezed states

Are there states that allow to break the $\frac{1}{n}$ scaling?

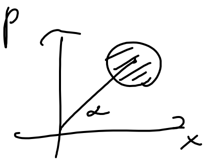
Intuition: when analysing phase estimation with coherent states we have seen that the problem lies in Poissonian fluctuations of photon count. It is known that there are squeezed states that in some settings may reveal sub-Poissonian photon number distribution.

- Squeezed states

$$[a, a^\dagger] = 1$$

$$\hat{x} = \frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}} \qquad \hat{p} = \frac{a - a^\dagger}{i\sqrt{2}} \qquad [x, p] = \frac{1}{2}\left(-\frac{1}{i} - \frac{1}{i}\right) = i$$

$$\Delta^2 x \, \Delta^2 p \geq \frac{1}{4}$$



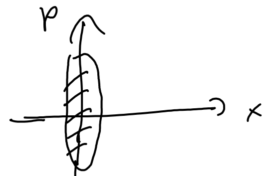
For a coherent state $\Delta^2_x = \Delta^2_p = \frac{1}{2}$

$$\langle\alpha| \tfrac{1}{2}(a + a^\dagger)^2 |\alpha\rangle = \tfrac{1}{2}\cdot(\alpha^2 + \alpha^{*2} + 2|\alpha|^2 + 1)$$

$$\langle\alpha| \tfrac{1}{\sqrt{2}}(a + a^\dagger)|\alpha\rangle = \tfrac{1}{\sqrt{2}}(\alpha + \alpha^*) \qquad \Delta^2_x = \frac{1}{2}$$

There are states with e.g. $\Delta p^2 < \frac{1}{2}$ $\Delta x^2 > \frac{1}{2}$ such that $\Delta x^2 \Delta p^2 = \frac{1}{4}$

Squeezed vacuum:



$$|r\rangle = S_r |vac\rangle,$$

$$S_r = e^{\frac{1}{2} r (a^2 - a^{\dagger 2})} \qquad \Delta^2 x = \frac{1}{2} e^{-2r} \qquad \Delta^2 p = \frac{1}{2} e^{2r}$$

$$\langle r| a^\dagger a |r\rangle = \langle vac| (a^\dagger \cosh r - a \sinh r)(a \cosh r - a^\dagger \sinh r)|vac\rangle = \sinh^2 r$$

$$\Delta^2 x = ? \quad \langle vac| e^{\frac{1}{2} r (a^{\dagger 2} - a^2)} \tfrac{1}{2}(a + a^\dagger)^2 e^{\frac{1}{2} r (a^2 - a^{\dagger 2})} |vac\rangle$$

$$e^{\frac{1}{2} r (a^{\dagger 2} - a^2)} a \, e^{\frac{1}{2} r (a^2 - a^{\dagger 2})} = a + \underbrace{\tfrac{1}{2} r [a^{\dagger 2}, a]}_{\substack{\frac{1}{2} r (-a^\dagger - a^\dagger) \\ = -r a^\dagger}} + \tfrac{1}{2}\left(\left[\tfrac{1}{2} r a^{\dagger 2}, -r a^\dagger\right]\right) = -\tfrac{1}{2} r^2 a \; \ldots$$

$$= a \cosh r - a^\dagger \sinh r$$

$$S_r^\dagger a S_r = a \cosh r - a^\dagger \sinh r$$

$$S_r^\dagger a^\dagger S_r = a^\dagger \cosh r - a \sinh r$$

$$\langle vac| \tfrac{1}{2}\cdot(a \cosh r - a^\dagger \sinh r + a^\dagger \cosh r - a \sinh r)^2 |vac\rangle =$$

$$= \tfrac{1}{2}\cdot(\cosh r^2 + \sinh^2 r - 2\cosh r \sinh r) = \tfrac{1}{2}\cdot(\cosh r - \sinh r)^2 =$$

$$= \tfrac{1}{2}\cdot(e^{-r})^2 = \tfrac{1}{2} e^{-2r}$$

Consider the following setup. [Caves 1981]

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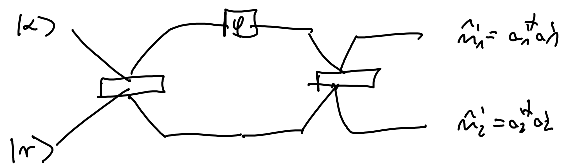


$$a_1' = \cos\frac{\varphi}{2}\, a_1 + i\sin\frac{\varphi}{2}\, a_2\,, \quad a_2' = \cos\frac{\varphi}{2}\, a_2 + i\sin\frac{\varphi}{2}\, a_1$$

$$\langle n_1\rangle = \langle\alpha|\langle r|\left(\cos\frac{\varphi}{2} a_1^\dagger - i\sin\frac{\varphi}{2} a_2^\dagger\right)\left(\cos\frac{\varphi}{2} a_1 + i\sin\frac{\varphi}{2} a_2\right)|\alpha\rangle|r\rangle =$$

$$= \cos^2\frac{\varphi}{2}\,|\alpha|^2 + \sin^2\frac{\varphi}{2}\,\sinh^2 r$$

$$\langle n_2\rangle = \langle\alpha|\langle r|\left(-i\sin\frac{\varphi}{2} a_1^\dagger + \cos\frac{\varphi}{2} a_2^\dagger\right)\left(i\sin\frac{\varphi}{2} a_1 + \cos\frac{\varphi}{2} a_2\right)|\alpha\rangle|r\rangle =$$

$$= \sin^2\frac{\varphi}{2}\,|\alpha|^2 + \cos^2\frac{\varphi}{2}\,\sinh^2 r$$

$$\langle n_1\rangle - \langle n_2\rangle = |\alpha|^2\cos\varphi + \sinh^2 r\cos\varphi = \cos\varphi\left(|\alpha|^2 - \sinh^2 r\right)$$

Estimator $\cos\tilde\varphi = \dfrac{n_1 - n_2}{|\alpha|^2 - \sinh^2 r}$

Now calculate the variance

$$\langle (n_1 - n_2)^2\rangle$$

$$\langle n_1^2\rangle = \cos^4\frac{\varphi}{2}\left(|\alpha|^4 + |\alpha|^2\right) + \sin^2\frac{\varphi}{2}\cos^2\frac{\varphi}{2}\,|\alpha|^2$$

$$\langle n_2^2\rangle = \sin^4\frac{\varphi}{2}\,.$$

$$\langle n_1^2\rangle + \langle n_2\rangle^2 - 2\langle n_1\rangle\langle n_2\rangle =$$

$$= \left(|\alpha|^4 + |\alpha|^2\right)\left(\cos^4\frac{\varphi}{2} + \sin^4\frac{\varphi}{2}\right)$$

$$+ 2\cos^2\frac{\varphi}{2}\sin^2\frac{\varphi}{2}|\alpha|^2 - 2\cos^2\frac{\varphi}{2}\sin^2\frac{\varphi}{2}|\alpha|^4$$

$$\delta^2_n =$$

$$\langle n_1^2\rangle = \langle\alpha, r|\left(\cos\frac{\varphi}{2} a_1^\dagger + i\sin\frac{\varphi}{2} a_2^\dagger\right)\left(\cos\frac{\varphi}{2} a_1 - i\sin\frac{\varphi}{2} a_2\right)\left(\cos\frac{\varphi}{2} a_1^\dagger + i\sin\frac{\varphi}{2} a_2^\dagger\right)\left(\cos\frac{\varphi}{2} a_1 - i\sin\frac{\varphi}{2} a_2\right)|\alpha r\rangle =$$

$$= \cos^4\frac{\varphi}{2}\left(|\alpha|^4 + |\alpha|^2\right) + \sin^4\frac{\varphi}{2}\langle r|\left(a_2^\dagger a_2\right)^2|r\rangle \; +$$

$$\cos^2\frac{\varphi}{2}\sin^2\frac{\varphi}{2}\cdot\Big(2|\alpha|^2\cdot\sinh^2 r + |\alpha|^2\underbrace{\langle r|a_2 a_2^\dagger|r\rangle}_{(1+\sinh^2 r)} + \sinh^2 r\underbrace{\langle\alpha|a_1 a_1^\dagger|\alpha\rangle}_{(1+|\alpha|^2)} - \left(\alpha^2 + \alpha^{*2}\right)\cosh r\sinh r\Big)$$

$$\langle r|\left(a_2^\dagger a_2\right)^2|r\rangle = \langle vac|\left[\left(\cosh r\, a_2^\dagger - \sinh r\, a_2\right)\left(\cosh r\, a_2 - \sinh r\, a_2^\dagger\right)\right]^2|vac\rangle$$

$$= \sinh^2 r\,\langle 1|\left(\cosh r\, a_2 - \sinh r\, a_2^\dagger\right)\left(\cosh r\, a_2^\dagger - \sinh r\, a_2\right)|1\rangle =$$

$$= \sinh^2 r\left(2\cosh^2 r + \sinh^2 r\right) = \sinh^2 r\left(2 + 3\sinh^2 r\right)$$

$$\langle r|a_2^2|r\rangle = \langle vac|\left(\cosh r\, a_2 - \sinh r\, a_2^\dagger\right)^2|vac\rangle = -\cosh r\sinh r$$

$$\langle n_1^2\rangle = \cos^4\frac{\varphi}{2}\left(|\alpha|^4 + |\alpha|^2\right) + \sin^4\frac{\varphi}{2}\sinh^2 r\left(2 + 3\sinh^2 r\right)$$

$$+ \cos^2\frac{\varphi}{2}\sin^2\frac{\varphi}{2}\cdot\left(4|\alpha|^2\sinh^2 r + |\alpha|^2 + \sinh^2 r + \left(\alpha^2 + \alpha^{*2}\right)\sinh r\cosh r\right)$$

$$\langle n_2^2\rangle = \cos^4\frac{\varphi}{2}\sinh^2 r\left(2 + 3\sinh^2 r\right) + \sin^4\frac{\varphi}{2}\left(|\alpha|^4 + |\alpha|^2\right)$$

$$+ \cos^2\frac{\varphi}{2}\sin^2\frac{\varphi}{2}\cdot\Big(4|\alpha|^2\sinh^2 r + \underbrace{|\alpha|^2 + \sinh^2 r}_{\bar n} + \left(\alpha^2 + \alpha^{*2}\right)\sinh r\cosh r\Big)$$

$$\langle n_1 n_2\rangle = \langle\alpha, r|\left(\cos\frac{\varphi}{2} a_1^\dagger + i\sin\frac{\varphi}{2} a_2^\dagger\right)\left(\cos\frac{\varphi}{2} a_1 - i\sin\frac{\varphi}{2} a_2\right)\left(i\sin\frac{\varphi}{2} a_1^\dagger + \cos\frac{\varphi}{2} a_2^\dagger\right)\left(-i\sin\frac{\varphi}{2} a_1 + \cos\frac{\varphi}{2} a_2\right)|\alpha r\rangle$$

$$= \cos^4\frac{\varphi}{2}\,|\alpha|^2\sinh^2 r + \sin^4\frac{\varphi}{2}\,|\alpha|^2\sinh^2 r$$

$$+ \cos^2\frac{\varphi}{2}\sin^2\frac{\varphi}{2}\cdot\Big(|\alpha|^4 + |\alpha|^2 \quad \sinh^2 r\left(2 + 3\sinh^2 r\right) - |\alpha|^2\left(1 + \sinh^2 r\right) - \sinh^2 r\left(|\alpha|^2 + 1\right)$$
$$- \left(\alpha^2 + \alpha^{*2}\right)\cosh r\sinh r\Big)$$

$$\langle (n_1 - n_2)^2\rangle = \cos^4\frac{\varphi}{2}\cdot\left(|\alpha|^4 + |\alpha|^2 + \sinh^2 r\left(2 + 3\sinh^2 r\right) - 2|\alpha|^2\sinh^2 r\right)$$

$$+\sin^4\frac{\varphi}{2}\cdot\left(|\alpha|^4+|\alpha|^2+\sinh^2 r(2+3\sinh^2 r)-2|\alpha|^2\sinh^2 r\right)$$

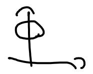
$$+2\sin^2\frac{\varphi}{2}\cos^2\frac{\varphi}{2}\cdot\left(4|\alpha|^2\sinh^2 r+2\bar{n}-|\alpha|^4-|\alpha|^2-\sinh^2 r(2+3\sinh^2 r)+2|\alpha|^2\sinh^2 r-2(\alpha^2+\alpha^{*2})\sinh r\cosh r\right)$$

$$=\cos^2\varphi\left(|\alpha|^4+|\alpha|^2+\sinh^2 r(2+3\sinh^2 r)-2|\alpha|^2\sinh^2 r\right)+\sin^2\varphi\left(2|\alpha|^2\sinh^2 r+\bar{n}+(\alpha^2+\alpha^{*2})\cosh r\sinh r\right)$$

$$\Delta^2(m_1-m_2)=\cos^2\varphi\left(\cancel{|\alpha|^4}+|\alpha|^2+\sinh^2 r(2+2\sinh^2 r)-\cancel{2|\alpha|^2\sinh^2 r}-\cancel{|\alpha|^4}+\cancel{2|\alpha|^2\sinh^2 r}-\cancel{\sinh^4 r}\right)$$

$$+\sin^2\varphi\left(2|\alpha|^2\sinh^2 r+\bar{n}+(\alpha^2+\alpha^{*2})\cosh r\sinh r\right)=$$

$\Big\}$ if $\alpha^{*2}=\alpha^2=-|\alpha|^2$



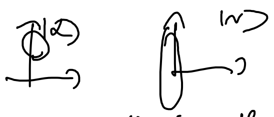
$$=|\alpha|^2\cos^2\varphi+2\cos^2\varphi\cosh^2 r\sinh^2 r+\left(2|\alpha|^2\cdot\frac{1}{2}\cdot(e^{-2r}-1)+|\alpha|^2+\sinh^2 r\right)$$

$$=|\alpha|^2\cos^2\varphi+2\cos^2\varphi\cosh r\sinh r+\sin^2\varphi\left(|\alpha|^2e^{-2r}+\sinh^2 r\right)$$

$$\Delta^2\cos\varphi=\frac{|\alpha|^2\cos^2\varphi+2\cos^2\varphi\cosh r\sinh r+\sin^2\varphi\left(|\alpha|^2e^{-2r}+\sinh^2 r\right)}{\left(|\alpha|^2-\sinh^2 r\right)^2}$$

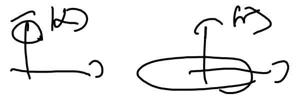
$$\Delta^2\varphi=\frac{\left(|\alpha|^2+2\cosh r\sinh r\right)\operatorname{ctg}^2\varphi+\left(|\alpha|^2e^{-2r}+\sinh^2 r\right)}{\left(|\alpha|^2-\sinh^2 r\right)^2}$$

Intuition:



since we measure quadratures $a_1\cos\frac{\varphi}{2}+ia_2\sin\frac{\varphi}{2}$ we have

$\sim$



smaller fluctuation in the direction α.

Optimal sensitivity is around $\varphi=\frac{\pi}{2}$

$$\Delta^2\varphi=\frac{|\alpha|^2e^{-2r}+\sinh^2 r}{\left(|\alpha|^2-\sinh^2 r\right)^2}$$

We fix $\bar{n}=|\alpha|^2+\sinh^2 r$

$$\Delta^2\varphi=\frac{\left(\bar{n}-\sinh^2 r\right)e^{-2r}+\sinh^2 r}{\left(\bar{n}-2\sinh^2 r\right)^2}$$

$\Big\}$ assume $|\alpha|^2\gg\sinh^2 r\gg 1$

$$\Delta^2\varphi=\frac{\bar{n}e^{-2r}+\frac{1}{4}e^{2r}}{\bar{n}^2}$$

$$\frac{d}{dx}\left(\bar{n}x+\frac{1}{4}\cdot\frac{1}{x}\right)=0$$

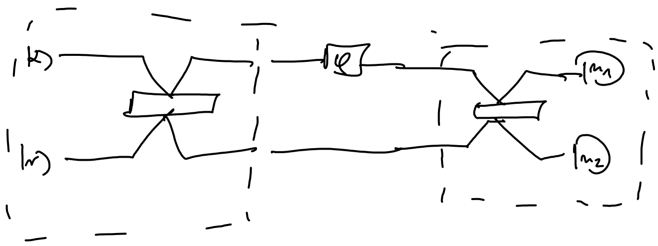
$$\bar{n}-\frac{1}{4x^2}=0\qquad x=\frac{1}{2\sqrt{\bar{n}}}$$

$$\Delta^2\varphi=\frac{\frac{\sqrt{\bar{n}}}{2}+\frac{1}{2}\sqrt{\bar{n}}}{\bar{n}^2}=\frac{1}{\bar{n}^{\frac{3}{2}}}$$

Better scaling. How far we can go?

6. Looking for the optimal estimation schemes

Three element to optimize over:



$$\tilde{\varphi} = \underset{\cdots}{\ldots} \arccos\left(\frac{m_1 - m_2}{|\alpha|^2 - \sinh^2 r}\right)$$

input state preparation — phase sensing — measurement — estimator

$|\psi_{in}\rangle$, $\mathcal{U}_\varphi = e^{i\varphi a^\dagger a}$, $\{\Pi_\xi\}$. $\tilde{\varphi}(m)$

$\rho_{in} = |\psi_{in}\rangle\langle\psi_{in}|$

general POVM measurement

$\Pi_\xi \geq 0, \int d\xi\, \Pi_\xi = \mathbb{1}$

Optimal phase estimation should minimize the average cost.

$C_{\tilde{\varphi},\varphi} = (\tilde{\varphi} - \varphi)^2$, or rather something periodic : $4\sin^2\left(\frac{\varphi - \tilde{\varphi}}{2}\right)$

Average cost :

$$\overline{\delta\varphi^2} = \overline{C} = \int d\varphi\, \underset{\substack{\uparrow \\ \text{a priori distribution}}}{p(\varphi)} \int d\xi\, p(\xi|\varphi)\, C_{\tilde{\varphi}(\xi),\varphi}$$

$$p(\xi|\varphi) = \mathrm{Tr}\big(\Pi_\xi \underbrace{\mathcal{U}_\varphi\, \rho_{in}\, \mathcal{U}_\varphi^\dagger}_{\rho_\varphi}\big)$$

problem : $\min\limits_{|\psi_{in}\rangle, \{\Pi_\xi\}, \tilde{\varphi}(\xi)} \overline{C}$ extremely hard

Two extreme ways to approach the problem :

— local approach $p(\varphi) \approx \delta(\varphi - \varphi_0)$ $\Big\{$ changes around

— global approach $p(\varphi) = \frac{1}{2\pi}$ sensing small

$\}$ a known phase

$\Big\{$ no apriori knowledge

7. Local approach { Maybe we can skip it if ... derive it

If we put $p(\varphi) = \delta(\varphi - \varphi_0)$ $\overline{C} = \int d\xi\, \mathrm{Tr}(\Pi_\xi\, \rho_{\varphi_0})\, C_{\tilde{\varphi}(\xi),\varphi_0}$

trivial way out $\Pi_{\tilde{\varphi} = \varphi_0} = \mathbb{1}$

to avoid trivial solution. We want to force the estimation to be sensitive to small changes of φ around φ_0

– „first order sensitivity".

We impose local-unbiasedness constraint

$\langle\tilde{\varphi}\rangle = \int d\xi\, \tilde{\varphi}(\xi)\, p(\xi|\varphi_0) = \varphi_0$ unbiasedness condition (too strong)

$$\left.\frac{d\langle\tilde{\varphi}\rangle}{d\varphi}\right|_{\varphi=\varphi_0} = 1 \qquad \int d\xi\, \tilde{\varphi}(\xi) \left.\frac{dp(\xi|\varphi)}{d\varphi}\right|_{\varphi=\varphi_0} = 1 \quad \text{locally unbiased condition}$$

$$\underbrace{\int d\xi\, p(\xi|\varphi_0)(\tilde{\varphi}(\xi) - \varphi_0)^2}_{\delta^2\tilde{\varphi}} \underbrace{\int d\xi\, \frac{1}{p(\xi|\varphi_0)}\left(\frac{dp(\xi|\varphi_0)}{d\varphi}\right)^2}_{F} \quad \text{C-S}$$

$$\int d\eta\, p(\eta|\varphi_0)\left(\tilde\varphi(\eta)-\varphi_0\right)^2 \int d\eta \frac{1}{p(\eta|\varphi_0)}\left(\frac{dp(\eta|\varphi_0)}{d\varphi_0}\right)^2 \quad \overset{C-S}{\geq}$$

$$\geq \int d\eta\, \left(\tilde\varphi(\eta)-\varphi_0\right)\sqrt{p(\eta|\varphi_0)}\,\frac{1}{\sqrt{p(\eta|\varphi_0)}}\,\frac{dp(\eta|\varphi_0)}{d\varphi_0} =$$

$$= \underbrace{\int d\eta\, \tilde\varphi(\eta)\frac{dp(\eta|\varphi_0)}{d\varphi_0}}_{1} - \underbrace{\varphi_0\int d\eta\, \frac{dp(\eta|\varphi_0)}{d\varphi_0}}_{0} = 1$$

$$\delta^2\tilde\varphi \cdot F \geq 1 \qquad \underbrace{\delta^2\tilde\varphi \geq \frac{1}{F}}_{\text{Cramer-Rao bound}}$$

We have got rid of the estimator problem.
Now we can just look at Fisher

$$F = \int d\eta \frac{1}{p(\eta|\varphi)}\left(\frac{dp(\eta|\varphi_0)}{d\varphi_0}\right)^2$$

and maximize F over $|\psi_{in}\rangle$, Π_η.

For k independent realization $F^{(k)} = k\cdot F$

$$\delta^2\varphi \geq \frac{1}{kF}$$

For $k\to\infty$ Max-likelihood estimator saturates C-R bound.

Notice that F tells just the local variations in $p(\eta|\varphi)$.

We may go further and get rid of the optimization over $\{\Pi_\eta\}$

$$F = \int d\eta \frac{1}{\mathrm{Tr}(\Pi_\eta \rho_\varphi)}\left(\frac{d(\mathrm{Tr}\,\Pi_\eta\rho_\varphi)}{d\varphi}\right)^2 = \int d\eta \frac{1}{\mathrm{Tr}(\Pi_\eta\rho_\varphi)}\left[\mathrm{Tr}\left(\Pi_\eta \frac{d\rho_\varphi}{d\varphi}\right)\right]^2$$

$$\left\{ \frac{d\rho_\varphi}{d\varphi} = \frac{1}{2}(\Lambda\rho_\varphi + \rho_\varphi\Lambda) \qquad \Lambda_{ij} = \frac{2}{p_i+p_j}\left(\frac{d\rho_\varphi}{d\varphi}\right)_{ij} \quad \text{in } \rho_\varphi \text{ eigenbasis} \right.$$

$$= \int d\eta \frac{\left(\mathrm{Tr}\left[\frac{1}{2}\Pi_\eta(\Lambda\rho_\varphi+\rho_\varphi\Lambda)\right]\right)^2}{\mathrm{Tr}(\Pi_\eta\rho_\varphi)} = \qquad \left\{ |\mathrm{Tr}A^\dagger B|^2 \leq \mathrm{Tr}A^\dagger A\cdot \mathrm{Tr}B^\dagger B \right.$$

$$= \int d\eta \frac{(\mathrm{Re}\,\mathrm{Tr}(\Pi_\eta\Lambda\rho_\varphi))^2}{\mathrm{Tr}(\Pi_\eta\rho_\varphi)} \leq \int d\eta \frac{|\mathrm{Tr}(\Pi_\eta\Lambda\rho_\varphi)|^2}{\mathrm{Tr}(\Pi_\eta\rho_\varphi)}$$

$$\left\{ A = \sqrt{\Pi_\eta}\sqrt{\rho_\varphi} \qquad B = \sqrt{\Pi_\eta}\Lambda\sqrt{\rho_\varphi} \qquad \mathrm{Tr}A^\dagger B = \mathrm{Tr}\sqrt{\rho_\varphi}\sqrt{\Pi_\eta}\sqrt{\Pi_\eta}\Lambda\sqrt{\rho_\varphi} = \right.$$
$$= \mathrm{Tr}(\rho_\varphi\Pi_\eta\Lambda)$$

$$\leq \int d\eta \frac{\mathrm{Tr}\,\rho_\varphi\Pi_\eta}{\mathrm{Tr}\,\rho_\varphi\Pi_\eta}\cdot \mathrm{Tr}\left(\sqrt{\rho_\varphi}\Lambda\sqrt{\Pi_\eta}\sqrt{\Pi_\eta}\Lambda\sqrt{\rho_\varphi}\right) = \mathrm{Tr}(\rho_\varphi\Lambda^2)$$

$$F_Q = \mathrm{Tr}(\rho_\varphi\Lambda^2) \qquad \underbrace{\delta^2\tilde\varphi \geq \frac{1}{F_Q}}_{\text{Q. C-R bound.}}$$

For pure states it is simple :

$$\frac{d\,|\psi_\varphi\rangle\langle\psi_\varphi|}{d\varphi} = |\psi_\varphi'\rangle\langle\psi_\varphi| + |\psi_\varphi\rangle\langle\psi_\varphi'|$$

so it is enought to choose $\Lambda = 2\big(|\psi_\varphi'\rangle\langle\psi_\varphi| + |\psi_\varphi\rangle\langle\psi_\varphi'|\big)$

$$\Lambda \cdot |\psi_\varphi\rangle\langle\psi_\varphi| = 2|\psi_\varphi'\rangle\langle\psi_\varphi| + 2|\psi_\varphi\rangle\langle\psi_\varphi'|\psi_\varphi\rangle\langle\psi_\varphi|$$

$$|\psi_\varphi\rangle\langle\psi_\varphi|\cdot\Lambda = 2|\psi_\varphi\rangle\langle\psi_\varphi|\psi_\varphi'\rangle\langle\psi_\varphi| + 2|\psi_\varphi\rangle\langle\psi_\varphi'|$$

$$F_Q = 4\Big(\langle\psi_\varphi|\big(|\psi_\varphi'\rangle\langle\psi_\varphi|\psi_\varphi'\rangle\langle\psi_\varphi| + |\psi_\varphi\rangle\langle\psi_\varphi'|\psi_\varphi\rangle\langle\psi_\varphi'| + |\psi_\varphi'\rangle\langle\psi_\varphi'| + \langle\psi_\varphi'|\psi_\varphi\rangle\big)|\psi_\varphi\rangle$$

$$= 4\cdot\big(\langle\psi_\varphi|\psi_\varphi'\rangle^2 + \langle\psi_\varphi'|\psi_\varphi\rangle^2 + |\langle\psi_\varphi|\psi_\varphi'\rangle|^2 + \langle\psi_\varphi'|\psi_\varphi'\rangle\big)$$

$\Big\{\ \langle\psi_\varphi|\psi_\varphi'\rangle + \langle\psi_\varphi'|\psi_\varphi\rangle = 0$

$$= 4\cdot\big(\langle\psi_\varphi'|\psi_\varphi'\rangle - |\langle\psi_\varphi|\psi_\varphi'\rangle|^2$$

Dependence only on $|\psi_{in}\rangle$!

Now we simply look for the state that maximizes F_Q. For phase estimation:

$$|\psi_\varphi'\rangle = \frac{d}{d\varphi} e^{i a^\dagger a \varphi} |\psi_{in}\rangle$$

$$F_Q = 4\cdot\big(\langle\psi_{in}|(a^\dagger a)^2|\psi_{in}\rangle - |\langle\psi_{in}|a^\dagger a|\psi_{in}\rangle|^2\big) =$$

$$= 4\cdot\delta^2 n_a$$

$$\delta\varphi^2\cdot\delta^2 n_a \geq \frac{1}{4}$$

Heisenberg like relation phase estimation uncertainty vs photon number uncertainty in "phase sensing" arm

The optimal state = the one that maximizes $\delta^2 n_a$

<u>Example</u> Find the optimal N photon state

$$|\psi\rangle = \sum_{n=0}^{N} \alpha_n |n, N-n\rangle$$

number of photons in a, b modes respectively

$$\langle \hat{n}_a^2\rangle - \langle\hat{n}_a\rangle^2 \leq \left(\frac{n_{max} - n_{min}}{2}\right)^2 = \frac{N^2}{4}$$

fact from linear algebra

$n_{max} = N \qquad n_{min} = 0$

$$|\psi_{in}\rangle = \frac{1}{\sqrt{2}}\cdot\big(|N,0\rangle + |0,N\rangle\big) \qquad \delta^2_n = \frac{N^2}{4}$$

$$\underbrace{\delta^2\varphi \geq \frac{1}{N^2}}$$ Heisenberg scaling

Interpretation: $|\psi_\varphi\rangle = \frac{1}{\sqrt{2}}\big(|N,0\rangle + e^{iN\varphi}|0,N\rangle\big)$

Interpretation: $|\psi_\varphi\rangle = \frac{1}{\sqrt{2}}\left(|N,0\rangle + e^{iN\varphi}|0,N\rangle\right)$

- N times faster oscillation than in single photon state, hence N-times increase in precision.

but for practical implementations useless, $\frac{2\pi}{N}$ ambiguity, we need to perform two-stage estimation.
- rough estimation
- more precise

If we are given total resource of N photons it requires some thought how to use them optimally ....

8. Global approach.

Since φ is just the label, we may as well label povms with estimated values $\tilde{\varphi}$. $\{\Pi_{\tilde{\varphi}}\}$

$$\int_0^{2\pi} \frac{d\tilde{\varphi}}{2\pi} \Pi_{\tilde{\varphi}} = \mathbb{1}$$

problem: $\min_{|\psi_{in}\rangle, \{\Pi_{\tilde{\varphi}}\}} \overline{C}$ still very difficult

$$\overline{C} = \int d\varphi\, p(\varphi) \int \frac{d\tilde{\varphi}}{2\pi} \mathrm{Tr}\left(\Pi_{\tilde{\varphi}}\ \rho_\varphi \right) C_{\tilde{\varphi},\varphi}$$

We may make use of the U_φ symmetry, no phase is distinguished. $p(\varphi) = \frac{1}{2\pi}$, $C_{\tilde{\varphi}+\varphi_0, \varphi+\varphi_0} = C_{\tilde{\varphi},\varphi}$

Optimal measurement: is covariant:

$$\Pi_{\tilde{\varphi}} = U_{\tilde{\varphi}} \Pi_0 U_{\tilde{\varphi}}^\dagger$$

$$\overline{C} = \int \frac{d\varphi}{2\pi}\frac{d\tilde{\varphi}}{2\pi} \mathrm{Tr}\left(U_{\tilde{\varphi}} \Pi_0 U_{\tilde{\varphi}}^\dagger U_\varphi \rho_{in} U_\varphi^\dagger \right) C_{\tilde{\varphi},\varphi} =$$

$$\begin{cases} \varphi' = \varphi - \tilde{\varphi} \\ \varphi'' = \tilde{\varphi} \end{cases}$$

$$= \int \frac{d\varphi'}{2\pi} \mathrm{Tr}\left(\Pi_0 U_{\varphi'} \rho_{in} U_{\varphi'}^\dagger \right) C_{0,\varphi'}$$

Covariant measurements - a detour

General problem.

$g \in G$ - group element $\rightarrow$ encoded via a unitary representation $|\psi_g\rangle = U_g |a\rangle$

In general could be in a mixed state

$$\rho_g = U_g \rho_0 U_g^\dagger$$

$\{\Pi_{\tilde g}\}$ – measurement result denotes the estimated value $\tilde g$

$C_{g,\tilde g}$ – cost function

$$\bar C = \int dg\, d\tilde g\, \mathrm{Tr}(\Pi_{\tilde g}\, \rho_g)\, C_{g,\tilde g}$$

Assumptions. (Estimation problem has symmetry with respect to G)

- dg – Haar measure of group G

$\{\quad g' = hg \qquad dg' = dg$

- $C_{hg,h\tilde g}$ – cost function left invariant

Notice all assumptions are trivially satisfied for global approach to phase estimation:

$G = U(1)$, $\quad U_{\varphi_1} U_{\varphi_2} = U_{\varphi_1+\varphi_2} \qquad d\varphi = d\varphi'$

$\varphi' = \varphi + \varphi_0$

$C_{\varphi,\tilde\varphi} = 4\sin^2\frac{\varphi-\tilde\varphi}{2} = C_{\varphi+\varphi_0,\,\tilde\varphi+\varphi_0}$ OK

Definition

$\{\Pi_{\tilde g}\}$ is covariant with respect to group

$G \iff \forall_{\tilde g,h}\; U_h \Pi_{\tilde g} U_h^\dagger = \Pi_{h\tilde g}$

Corollary

If $\{\Pi_{\tilde g}\}$ is covariant

$\Pi_{\tilde g} = U_{\tilde g}\, \Pi_e\, U_{\tilde g}^\dagger$

The measurement is fully defined with a single operator Π_e

Theorem

If the estimation problem has symmetry with respect to group G then the optimal measurement can always be found among measurements covariant with respect to G

Proof

Let $\Pi_{\tilde{g}}^{opt}$ be the optimal measurement

, minimizing $\bar{C}$:

$$\bar{C}_{opt} = \int dg\, d\tilde{g}\, \mathrm{Tr}\left(\Pi_{\tilde{g}}^{opt} \rho_g\right) C_{g,\tilde{g}}$$

Define

$$\Pi_{\tilde{g}}^{cov} = \int dg'\, U_{g'}^{\dagger} \Pi_{g'\tilde{g}}^{opt} U_{g'}$$

Measurement $\Pi_{\tilde{g}}^{cov}$ is indeed covariant

$$U_h \Pi_{\tilde{g}}^{cov} U_h^{\dagger} = \int dg'\, U_{hg'^{-1}} \Pi_{g'\tilde{g}}^{opt} U_{g'h^{-1}}$$

$$\underline{g' \to g'h} \quad \int dg'\, U_{g'} \Pi_{g'h\tilde{g}}^{opt} U_{g'} = \Pi_{h\tilde{g}}^{cov}$$

And gives the same cost as $\Pi_{\tilde{g}}^{opt}$:

$$\bar{C}_{cov} = \int dg\, d\tilde{g}\, \mathrm{Tr}\left(\Pi_{\tilde{g}}^{cov} \rho_g\right) C_{\tilde{g},g} =$$

$$= \int dg\, d\tilde{g}\, \mathrm{Tr}\left(\int dg'\, U_{g'}^{\dagger} \Pi_{g'\tilde{g}}^{opt} U_{g'} U_g \rho_e U_g^{\dagger}\right) C_{g,\tilde{g}}$$

$$= \int dg\, d\tilde{g}\, dg'\, \mathrm{Tr}\left(U_{g'g}^{\dagger} \Pi_{g'\tilde{g}}^{opt} U_{g'g} \rho_e\right) C_{g,\tilde{g}}$$

$$\begin{cases} g \to g'^{-1} g \\ \tilde{g} \to g'^{-1} \tilde{g} \end{cases}$$

$$= \int dg\, d\tilde{g}\, dg'\, \mathrm{Tr}\left(U_g^{\dagger} \Pi_{\tilde{g}}^{opt} U_g \rho_e\right) C_{g'^{-1}g,\, g'^{-1}\tilde{g}} =$$

$$= \int dg\, d\tilde{g}\, dg'\, \mathrm{Tr}\left(\Pi_{\tilde{g}}^{opt} \rho_g\right) C_{g,\tilde{g}} = C_{opt} \qquad \blacksquare$$

Problem can be simplified

Problem can be simplified

$$\overline{C} = \int dg\, d\tilde{g}\ \mathrm{Tr}(\Pi_{\tilde{g}}\, \rho_g)\, C_{g,\tilde{g}} =$$

$$= \int dg\, d\tilde{g}\ \mathrm{Tr}\left(U_{\tilde{g}}\, \Pi_e\, U_{\tilde{g}}^{\dagger}\, U_g\, \rho_e\, U_g^{\dagger} \right) C_{g,\tilde{g}}$$

$$= \int dg\, d\tilde{g} \left(\mathrm{Tr}\left(U^{\dagger}_{\tilde{g}^{-1}g}\ \Pi_e\ U_{\tilde{g}^{-1}g}\ \rho_e \right) C_{g,\tilde{g}}\right.$$

$$\Big\{\ g \to \tilde{g} g$$

$$= \int dg\, d\tilde{g}\ \mathrm{Tr}\left(U_g^{\dagger}\, \Pi_e\, U_g\ \rho_e\right) C_{g,e}$$

$$\overline{C} = \int dg\ \mathrm{Tr}(\Pi_e\, \rho_g)\, C_{g,e}$$

<u>Find form of the problem</u>

$$\min_{|\psi_{in}\rangle,\ \Pi_e \geq 0} \overline{C}$$

$$\int dg\, U_g \Pi_e U_g^{\dagger} = \mathbb{1}$$

we optimize only over one operator !

$$\overline{C} = \int dg\ \mathrm{Tr}(\Pi_e\, \rho_g)\, C_{g,e}$$

<u>Return to phase estimation problem</u>

$$\overline{C} = \int \frac{d\varphi}{2\pi}\ \mathrm{Tr}\left(\Pi_0\ U_\varphi |\psi_{in}\rangle\langle\psi_{in}| U_\varphi^{\dagger}\right) C_{\varphi,0}$$

$$|\psi_{in}\rangle = \sum_m \alpha_m |m, N-m\rangle =: \sum_m \alpha_m |m\rangle$$

$$\overline{C} = \mathrm{Tr}\left(\Pi_0 \int \frac{d\varphi}{2\pi} \sum_{n,m} e^{i(n-m)\varphi}\, \alpha_n \alpha_m^*\, C_{\varphi,0}\, |n\rangle\langle m| \right)$$

$$= \sum_{n,m} \langle m|\Pi_0|n\rangle\ \alpha_n \alpha_m^* \int \frac{d\varphi}{2\pi}\, e^{i(n-m)\varphi}\, C_{\varphi,0}$$

$$\Big\{\ C_{\varphi,0} = 4\sin^2\frac{\varphi}{2} = 4\left(\frac{e^{\frac{i\varphi}{2}} - e^{-\frac{i\varphi}{2}}}{2i}\right)^2 = 2 - e^{i\varphi} - e^{-i\varphi}$$

$$= 2\sum_n \langle n|\Pi_0|n\rangle - \sum_n \langle n+1|\Pi_0|n\rangle \alpha_n \alpha^*_{n+1} - \sum_n \langle n|\Pi_0|n+1\rangle \alpha^*_n \alpha_{n+1}$$

Constraints on Π_c: $\Pi_c \geq 0$

$$\int U_\varphi \Pi_0 U_\varphi^\dagger \, d\varphi = \mathbb{1}$$

$$\Pi_0 = \sum_{n,m} \Pi_{n,m} |m\rangle\langle n| \qquad \int \frac{d\varphi}{2\pi} U_\varphi \Pi_c U_\varphi^\dagger = \sum \Pi_{n,m} |n\rangle\langle m| \int \frac{d\varphi}{2\pi} e^{i(n-m)\varphi}$$

$$= \sum_n \Pi_{n,n} |n\rangle\langle n| = \mathbb{1} \quad \Rightarrow \quad \langle n|\Pi_0|n\rangle = 1$$

$$\bar{C} = 2 - 2\,\mathrm{Re} \sum_n \langle n+1|\Pi_0|n\rangle \alpha_n \alpha^*_{n+1} \geq$$

$$\geq 2 - 2\sum_n |\langle n+1|\Pi_0|n\rangle|\, |\alpha_n||\alpha_{n+1}|$$

since $\Pi_c \geq 0$ and diagonal elements are 1 we know that $|\langle n+1|\Pi_c|n\rangle| \leq 1$ so

$$\bar{C} \geq 2 - 2\sum_n |\alpha_n||\alpha_{n+1}|$$

Moreover we can saturate this bound by choosing: $\alpha_n \geq 0$

$$\Pi_c = |e\rangle\langle e|$$

where $|e\rangle = \sum_{n=0}^{N} |n, N-n\rangle$ we have the optimal measurement

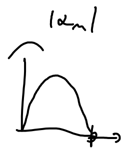
What is the optimal state:

$$\bar{C} = 2 - \langle\psi| \begin{bmatrix} 0 & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & 1 & 0 & 1 & \\ & & & 1 & 0 & 1 \\ & & & & 1 & 0 \end{bmatrix} |\psi\rangle$$

We look for eigenvector corresponding to maximal eigenvalue: solution is analytical

$$\bar{C}_{min} = 2\left(1 - \cos\left(\frac{\pi}{N+2}\right)\right)$$

$$|\psi_{in}\rangle = \sqrt{\frac{2}{N+2}} \sum_{n=0}^{N} \sin\frac{(n+1)\pi}{N+2} \, |n, N-n\rangle$$



[Berry, Wiseman 2000]

$$\delta\varphi^L = 2\left(1 - \cos\frac{\pi}{N+2}\right) \approx \frac{\pi^2}{(N+2)^2}$$

$$\delta\varphi^{1} = 2\left(1 - \cos\frac{\pi}{N+2}\right) \approx \frac{\pi^2}{(N+2)^2}$$

Heisenberg scaling

Completely different state than in local approach
here we are really sure we could in
practical implementation reach the Heisenberg
if we knew how to implement the
optimal measurement.