Lecture 2

10 sierpnia 2011
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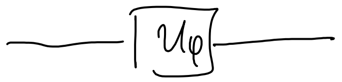
9. A Bigger picture on quantum metrology
[Giovannetti, Lloyd, Maccone 2006]

Consider a probe system (e.g. photon) which experiences $\mathcal{U}_\varphi = e^{i\varphi \hat{G}}$

$\hat{G}$ - generator of a phase shift



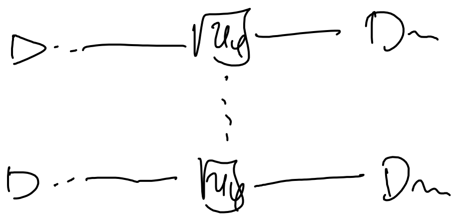
C-R bound

$$\delta\varphi\, \delta G \geq \frac{1}{2}$$

$$\delta G \leq \frac{(\lambda_+ - \lambda_-)}{2}$$

$$\left\{ \begin{array}{l} |\psi\rangle = \frac{1}{\sqrt{2}}\left(|e_{max}\rangle + |e_{min}\rangle\right) \\ \delta^2 G = \frac{1}{2}(\lambda_+^2 + \lambda_-^2) - \frac{1}{4}(\lambda_+ + \lambda_-)^2 = \\ \quad = \frac{1}{4}(\lambda_+ - \lambda_-)^2 \end{array} \right.$$

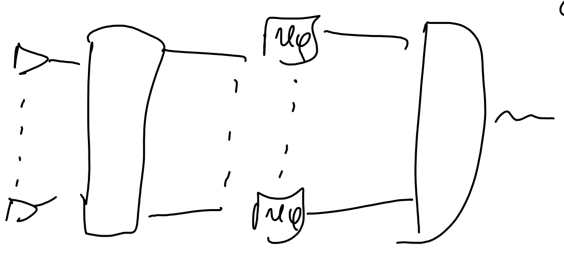
If we use N probes independently



$$\delta\varphi \geq \frac{1}{\sqrt{N}(\lambda_+ - \lambda_-)}$$

this corresponds to e.g. N single photons sent through the interferometer.

If we allow entangled input states and arbitrary measurement



$$\mathcal{U}_\varphi^{\otimes N} |\psi_N\rangle \qquad G^{(N)} = \sum_{i=1}^{N} G_i$$

$$\delta\varphi\, \delta G^{(N)} \geq \frac{1}{2}$$

$$\lambda_+^{(N)} = N \cdot \lambda_+ \qquad \lambda_-^{(N)} = N\lambda_-$$

$$\delta\varphi \geq \frac{1}{N(\lambda_+ - \lambda_-)}$$

Remark: local measurement are sufficient to reach the bounds

<u>Example</u>: Frequency standards

Cs fountain: two level atoms

$\omega_0 \updownarrow \frac{\;|1\rangle}{\;|0\rangle}$

$$|0\rangle^{\otimes N} \xrightarrow{\text{pulse}} \left(\tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right)^{\otimes N} \longrightarrow \left(\tfrac{1}{\sqrt{2}}\left(|0\rangle + e^{-i\Delta t}|1\rangle\right)\right)^{\otimes N} \longrightarrow \tfrac{1}{2}\cdot\left((1+e^{-i\Delta t})|0\rangle + (1-e^{-i\Delta t})|1\rangle\right)^{\otimes N}$$

$\Delta = \omega - \omega_0$ - frequency detuning
t - time of flight

exactly the same mathematical structure as in M-Z interferometer (Ramsey interferometry) now $\varphi \to \Delta \cdot t$

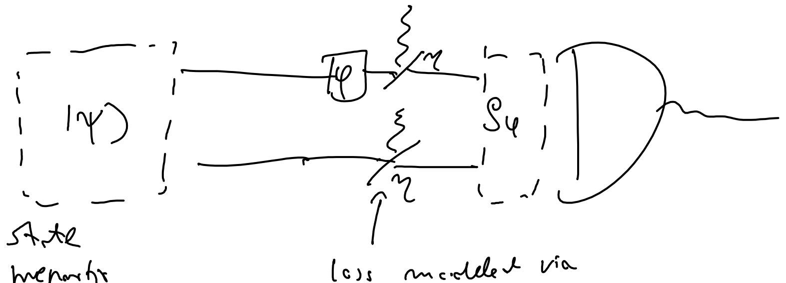
If we use independent atoms $\delta\Delta \geq \frac{1}{t\sqrt{N}}$

Using entangled states (spin squeezed states) could possibly lead to $\delta\Delta \geq \frac{1}{tN}$..

10. Role of decoherence

For practical implementations we need to take into account decoherence. Most common models

- photon loss (in Mach-Zehnder interferometer)
- dephasing (in Ramsey interferometry).



loss modelled via
fictitious beam-splitters
with transmission η

Mathematically problem more difficult since we deal with mixed states at the output

$$|n_a, n_b\rangle \longrightarrow \sum_{l_a=0}^{n_a}\sum_{l_b=0}^{n_b} \underbrace{\sqrt{\binom{n_a}{l_a}\eta_a^{n_a-l_a}(1-\eta_a)^{l_a}}}_{B^{n_a}_{l_a}(\eta_a)} \cdot \underbrace{\sqrt{\binom{n_b}{l_b}\eta_b^{n_b-l_b}(1-\eta_b)^{l_b}}}_{B^{n_b}_{l_b}(\eta_b)} \cdot$$

$$\cdot\, |n_a - l_a, n_b - l_b, \underbrace{l_a, l_b}_{\text{loss modes}}\rangle$$

$$|\psi\rangle = \sum_n \alpha_n |n, N-n\rangle \xrightarrow{\varphi} \sum_n \alpha_n e^{in\varphi}|n, N-n\rangle$$

$$\xrightarrow{\text{loss}} \sum \alpha_n e^{in\varphi} \sum^{n} \sum^{N-n} \sqrt{B^n_{l_a}(\eta_a)} \cdot \sqrt{B^{N-n}_{l_b}(\eta_b)} \cdot$$

$$\xrightarrow{\text{lose}} \sum_n \alpha_n e^{in\varphi} \sum_{l_a=0}^{n} \sum_{l_b=0}^{N-n} \sqrt{B^n_{l_a}(\eta_a)} \cdot \sqrt{B^{N-n}_{l_b}(\eta_b)} \cdot$$

$$\cdot |n-l_a, N-n-l_b, l_a, l_b\rangle =: |\underline{\Phi}\rangle$$

$$\rho_\varphi = \mathrm{Tr}_{l_a, l_b} |\underline{\Phi}^\varphi\rangle\langle\Phi^\varphi| =$$

$$= \sum_{l_a=0}^{N} \sum_{l_b=0}^{N-l_a} |\Psi^\varphi_{l_a,l_b}\rangle\langle\Psi^\varphi_{l_a,l_b}|$$

conditional state (unnormalized) provided l_a and l_b photons were lost

$$|\Psi^\varphi_{l_a,l_b}\rangle = \sum_{n=l_a}^{N-l_b} \alpha_n e^{in\varphi} \sqrt{B^n_{l_a}(\eta_a) B^{N-n}_{l_b}(\eta_b)} |n-l_a, N-n-l_b\rangle$$

Alternatively we may represent the channel using the Kraus representation. :

$$\rho_\varphi = \sum_{l_a=0}^{N} \sum_{l_b=0}^{N-l} K^\varphi_{l_a,l_b} |\psi\rangle\langle\psi| K^{\dagger\varphi}_{l_a l_b}$$

$$K^\varphi_{l_a,l_b} = (K_{l_a} \otimes K_{l_b}) \cdot U_\varphi$$

$$K_{l_a} = \sum_{n=l_a} \sqrt{B^n_{l_a}(\eta_a)} |n-l_a\rangle\langle n| , \quad K_{l_b} = \sum_{n=l_b} \sqrt{B^n_{l_b}(\eta_b)} = |n-l_b\rangle\langle n|$$

$$\sum_{l_a} K^\dagger_{l_a} K_{l_a} = \sum_{l_a} \sum_{n=l_a} |n\rangle\langle n| \, B^n_{l_a}(\eta_a) = \mathbb{1} .$$

To find the fundamental bounds we need to either calculate F_Q (local approach) or design optimal covariant estimation scheme (global approach)

Most interesting question. Do we still have qualitative precision enhancement i.e. $\frac{1}{N}$ (or $\frac{1}{N^\alpha}, \alpha > \frac{1}{2}$) instead of $\frac{1}{\sqrt{N}}$?

11. Bounds in the local approach in presence of

decoherence

In general it is impossible to write analytical formula for F_Q since for mixed states $F_Q = \mathrm{Tr}(\rho_\varphi \Lambda^2)$. It is even more unlikely to be possible to perform optimization over input states analytically.

Therefore we need to find more tractable bounds..

[Escher, Filho, Davidovich 2011]

General setup:

$$\rho_\varphi = \sum_i K_i^\varphi |\psi\rangle\langle\psi| K_i^{\varphi\dagger}$$

We can always look at it as a unitary transformation in an extended space S+E

$$\rho_\varphi = \mathrm{Tr}_E |\Phi^\varphi\rangle\langle\Phi^\varphi|$$

$$|\Phi^\varphi\rangle = U^\varphi_{S,E} |\psi\rangle_S \otimes |0\rangle_E = \sum_i K_i^\varphi \otimes \mathbb{1} |\psi\rangle \otimes |i\rangle$$

Kraus representation is not unique. We have freedom to apply local unitary V on the E subsystem. This is equivalent to new Kraus representation $K_i' = V_{i'i} K_i$.

Intuitive fact: Tracing out E can only reduce the information available on φ

$$F_Q(\rho_\varphi) \leq \underbrace{F_Q(|\Phi^\varphi\rangle)}_{\text{easy to calculate}}$$

More formally

$$F_Q(\rho_\varphi) = \max_{\Pi_m^S} \underbrace{F(\rho_\varphi, \Pi_m^S)}_{\text{classical Fisher}} = \max_{\{\Pi^S\}} F(|\Phi^\varphi\rangle, \Pi_m^S \otimes \mathbb{1}^E)$$

$$\leq \max F(|\Phi^\varphi\rangle, \Pi^{S,E}) = F_Q(|\Phi^\varphi\rangle)$$

$\{\Pi^{S,E}\}$

$$F_Q(|\varphi^\varphi\rangle) = 4\left(\langle \varphi^{\varphi'} | \varphi^{\varphi'}\rangle - |\langle \varphi^\varphi | \varphi^{\varphi'}\rangle|^2\right)$$

$$|\varphi^{\varphi'}\rangle = \sum_i \frac{dK_i^\varphi}{d\varphi} \otimes \mathbb{1}\, |\psi\rangle \otimes |i\rangle$$

$$F_Q(|\varphi^\varphi\rangle) = 4\cdot\Big(\langle\psi| \overbrace{\sum_i \frac{dK_i^{\varphi\dagger}}{d\varphi} \frac{dK_i^\varphi}{d\varphi}}^{H_2} |\psi\rangle - \Big|\langle\psi| \underbrace{\sum_i \frac{dK_i^{\varphi\dagger}}{d\varphi} K_i^\varphi}_{H_1} |\psi\rangle\Big|^2 \Big)$$

Is it useful?

<u>Theorem:</u>

$$F_Q(\rho_\varphi) = \min_{\{K_i^\varphi\}} F_Q(|\varphi^\varphi\rangle)$$

Proof based on the fact that second order expansion of Bures fidelity corresponds to F_Q:

$$F_B(\rho_\varphi, \rho_{\varphi+d\varphi}) = \left[\mathrm{Tr}\left(\sqrt{\rho_\varphi^{\frac{1}{2}} \rho_{\varphi+d\varphi} \rho_\varphi^{\frac{1}{2}}}\right)\right]^2 \approx$$

$$\approx 1 - \frac{\delta x^2}{4} \cdot F_Q(\rho_\varphi)$$

$$F_B(\rho_\varphi, \rho_{\varphi+d\varphi}) = \max_{|\psi_\varphi\rangle, |\psi_{\varphi+d\varphi}\rangle} |\langle\psi_\varphi|\psi_{\varphi+d\varphi}\rangle|^2$$

Intuition: there is always a purification in which access to environment is not helpful in estimating φ.

<u>Example:</u> Interferometry with loss

For simplicity only in one arm

• Let us take Kraus decomposition

$$K_l^{\varphi} = (K_l \otimes \mathbb{1}) \cdot U_\varphi$$

$$\frac{dK_l^\varphi}{d\varphi} = i(K_l \otimes \mathbb{1})\, a^\dagger a\, U_\varphi$$

$$\langle\psi| \sum_{l=0}^{\infty} \frac{dK_l^{\varphi\dagger}}{d\varphi} \frac{dK_l^\varphi}{d\varphi} |\psi\rangle = \langle\psi| U_\varphi^\dagger a^\dagger a \underbrace{\sum_l K_l^{\varphi\dagger} K_l^\varphi}_{\mathbb{1}} \, a^\dagger a\, U_\varphi |\psi\rangle$$

$$= \langle\psi| U_\varphi^\dagger (a^\dagger a)^2 U_\varphi |\psi\rangle = \langle\psi| (a^\dagger a)^2 |\psi\rangle$$

similarly for $\langle\psi| \sum_{l=0}^{\infty} \frac{dK_l^{\dagger}}{d\varphi} K_l |\psi\rangle = \langle\psi| a^\dagger a |\psi\rangle$

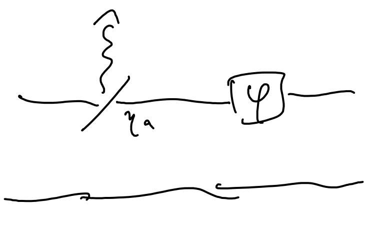
The same as in lossless case,
this result is useless !
It just tells us that Fisher ~~in~~ lossy
case will be less then Fisher in lossless case

• Let us take another Kraus representation
for a lossy interferometer

$$K_l'^{\varphi} = e^{-il a\varphi} K_l^\varphi$$

It is exactly the same as if we
change the order of U_φ and K_{ln}

$$K_l'^{\varphi} = U_\varphi (K_l \otimes \mathbb{1})$$ } first loss then phase sensing



It is obvious that environment „knows nothing" about the phase φ,
So this is a good try.
We should not be sure, however, that
we will ge a strict bound. Monitoring
E may still be helpful as we may make

E may still be helpful as we may make use of coherences that are lost if we look only at $S_\cdot$

$$\frac{d K_l'^{\varphi}}{d\varphi} = i a^\dagger a\, U_\varphi (K_l \otimes 1)$$

$$\langle\psi|\sum_l K_l^\dagger U_\varphi (a^\dagger a)^2 U_\varphi^\dagger K_l|\psi\rangle =$$

$$= \langle\psi| \sum_l \sum_{n=l} \sqrt{B_l^n(\eta)}\,|n\rangle\langle n-l|\,(a^\dagger a)^2 \sum_{n'=l} |n'-l\rangle\langle n'|\sqrt{B_l^{n'}(\eta)}|\psi\rangle$$

$$n = n'$$

$$= \langle\psi|\,\hat{n}_a^2\,|\psi\rangle - \langle\psi|\sum_{n=0}\sum_{l=0}^{n} |n\rangle\langle n|\, 2nl\, B_l^n(\eta)|\psi\rangle$$

$$+ \langle\psi|\sum_{n,l} |n\rangle\langle n|\, l^2\, B_l^n(\eta)\,|\psi\rangle =$$

$$\sum_{l=0}^{n} l\, B_l^n(\eta) = \sum_{l=0}^{n} l \binom{n}{l} \eta^{n-l}(1-\eta)^l = \sum_l \frac{n!}{(l-1)!(n-l)}\eta^{n-l}(1-\eta)^l =$$

$$= n(1-\eta)$$

$$\sum_{l=0} l^2 B_l^n(\eta) = \sum_l (1-\eta)\, l \cdot n \frac{(n-1)!}{(l-1)!(n-l)!}\eta^{n-l}(1-\eta)^{l-1} =$$

$$= + (1-\eta)n + (n-1)n(1-\eta)^2$$

$$\langle H_2\rangle = \langle n_a^2\rangle - 2(1-\eta)\langle n_a^2\rangle + (1-\eta)\langle n_a\rangle + (1-\eta)^2(\langle n_a^2\rangle - \langle n_a\rangle)$$

$$\langle H_1\rangle = \langle n\rangle - (1-\eta)\langle n\rangle = \eta\langle n\rangle$$

$$F_Q = 4\Big(\langle n^2\rangle\cdot(1-2+2\eta+(1-\eta)^2) + \langle n\rangle\cdot(1-\eta-(1-\eta)^2)$$
$$- \langle n\rangle^2\eta^2\Big)$$

$$= 4\eta^2\langle\Delta n^2\rangle + 4\eta(1-\eta)\langle n\rangle$$

A bit better bound but still would suggest that $\frac{1}{N}$ scaling survives

But we could use more general Kraus representation

$$K_l'^{\varphi} = e^{-i\alpha l\varphi} K_l^{\varphi} \qquad \text{then}$$

?

$$F_Q = 4\cdot\left(1-(1-\eta)\alpha\right)^2\langle\Delta n^2\rangle + 4\eta(1-\eta)\alpha^2\langle n\rangle$$

If we now take $\alpha = \frac{1}{1-\eta}$ then

$$F_Q = \frac{4\eta}{1-\eta}\langle n\rangle$$

Scales only linearly in the number of photons ☹

So precision will scale $\delta\varphi \gtrsim \sqrt{\frac{1-\eta}{4\eta N}}$

We lose the Heisenberg scaling ☹

12 Bounds in the global approach in the presence of loss (again for simplicity $\eta_G = 1$)

We can still use covariant measurement

$$\overline{C} = \int \frac{d\varphi}{2\pi} \mathrm{Tr}\left(\Pi_c\, \rho_\varphi\right) C_{\varphi,0}$$

$$\rho_\varphi = \mathrm{Tr}_l\ |\Phi^\varphi\rangle\langle\Phi^\varphi| =$$

$$= \sum_{l=0}^{N} |\psi_l^\varphi\rangle\langle\psi_l^\varphi|$$

$$|\psi_l^\varphi\rangle = \sum_{n=l}^{N} \alpha_n e^{in\varphi}\sqrt{B_l^n(\eta_a)}\,|n-l, N-n\rangle$$

One can argue that the optimal choice are $\alpha_n \in \mathbb{R}$ and $\Pi_c = |e\rangle\langle e|$

$$e = \sum_n |n, N-n\rangle$$

So finally:

$$\overline{C} = 2 - \langle\psi|\, A\, |\psi\rangle \qquad A = \begin{bmatrix} a & a & & & \\ a & a & a & & \\ & a & a & a & \\ & & a & a & a \\ & & & a & a \end{bmatrix}$$

$n-1$

$$A_{n-1,n} = A_{n,n-1} = \sum_{l=0} \sqrt{B_l^n(\eta_a) B_l^{n-1}(\eta_a)}$$

Looking for maximal eigenvalue is no longer possible analytically ....

<u>Fact:</u>

$$\lambda_{max} \leq \lambda_{max}'$$

wher λ_{max}' is the maximum eigenvalue of matrix A' where all element $A_{n-1,n}$ are replaced by the maximal one $A_{N-1,N}$

<u>Proof</u>:

For a matrix with all entries ≥ 0 eigenvector corresponding to λ_{max} has $|v\rangle = \sum \alpha_n |n\rangle$ where all $\alpha_n \geq 0$

$$\lambda_{max} = \langle v|A|v\rangle \leq \langle v|A'|v\rangle \leq \lambda'_{max} \quad \boxed{}$$

$$\bar{C} \geq 2 - 2A_{N-1,N} \cdot \cos\left(\frac{\pi}{N+2}\right) =$$

$$= 2\left[1 - \cos\left(\frac{\pi}{N+2}\right) \cdot \sum_{l=0}^{N-1} \sqrt{B_l^N(\eta) B_l^{N-1}(\eta)}\right]$$

expanding in $\frac{1}{N}$

$$\bar{C} \gtrsim 2\left[1 - \left(1 - \underbrace{\frac{\pi^2}{2(N+2)^2}}_{\text{not relevant}}\right) \cdot \left(1 - \frac{1-\eta}{8\eta N} + \dots\right)\right] = \frac{1-\eta}{4\eta N}$$

$$\delta\varphi \gtrsim \sqrt{\frac{1-\eta}{4\eta N}}$$

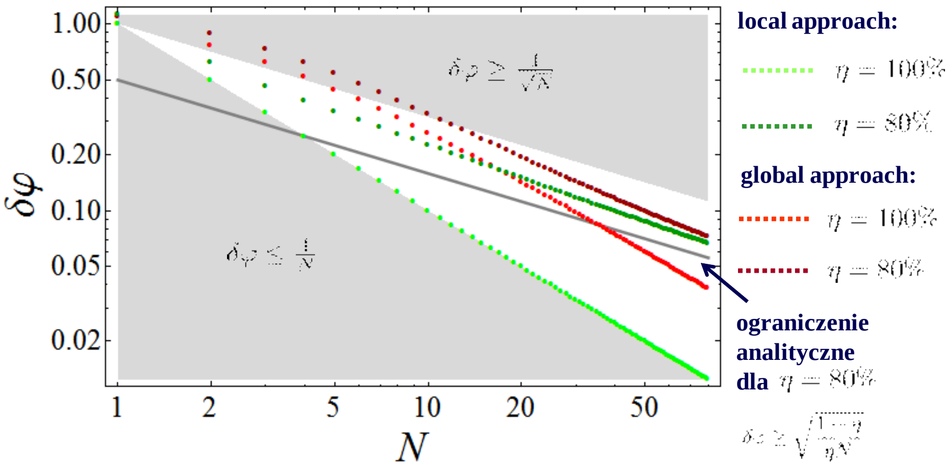
The same bound as in the local approach !

For losses in both arms, one can derive:

$$\delta\varphi \underset{\eta}{\gtrsim} \sqrt{\frac{1-\eta}{N}}$$

Summary in a plot :

Summary in a plot:



12. Outlook

- Is this behaviour typical in all relevant decoherence models
- looking for practical applications