Reconstruction of Quasi-Probability Distributions

I. Quantum Effects of a Single Oscillator

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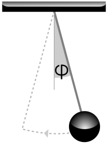
Contents

Nonclassicality

- Coherent states $|\alpha\rangle$: classical behavior
- Mixture of classical states:

$$\hat{\rho}_{\rm cl} = \sum_i p_i |\alpha_i\rangle\langle\alpha_i| \Rightarrow \int dP_{\rm cl}(\alpha)|\alpha\rangle\langle\alpha|$$

- General quantum state:[1] $\hat{\rho} = \int dP(\alpha)|\alpha\rangle\langle\alpha|$
- $P(\alpha) \cong$ quasiprobability: $P(\alpha) \neq P_{\rm cl}(\alpha)$



$P(\alpha)$ is often strongly singular! Experimental determination?

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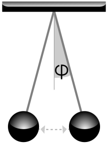

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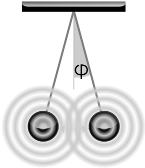

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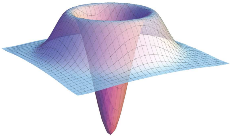

Experimental P function

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Entanglement

- Uncorrelated (product) states: $|a, b\rangle \equiv |a\rangle \otimes |b\rangle$
- Mixture of uncorrelated states $\Rightarrow$ separable states $\hat{\sigma}$:[2]

  $\hat{\sigma} = \sum_i p_i |a_i, b_i\rangle\langle a_i, b_i|$ (p_i: probability)

  $\Rightarrow \int dP_{\rm cl}(a, b)|a, b\rangle\langle a, b|$ ($P_{\rm cl}$: joint probability)
- General state: $\hat{\rho} = \int dP(a, b)|a, b\rangle\langle a, b|$
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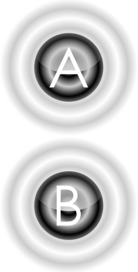

$P(a, b)$ is not unique! Identification of entanglement?

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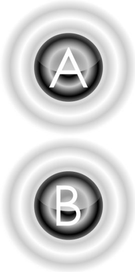

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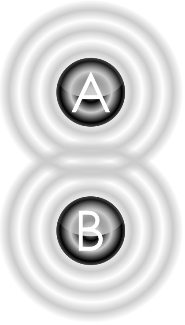

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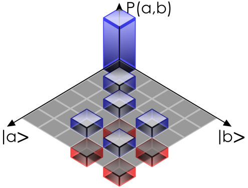

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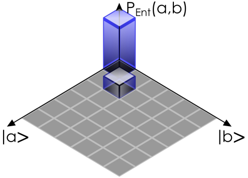

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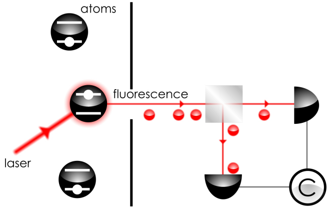
Quantum-field correlations

- Photon antibunching[3]
- Violation of Schwarz inequality:
  $\langle \mathcal{T}{:}\hat{I}(0)\hat{I}(\tau){:}\rangle > \langle{:}\,[\hat{I}(0)]^2\,{:}\rangle$
- Field operators:
  $\hat{E}^{(\pm)}(i) \equiv \hat{E}^{(\pm)}(\mathbf{r}_i, t_i)$
- P function $\Rightarrow$ P functional:
  $P(\alpha) \Rightarrow P(\{E^{(+)}(i)\})$



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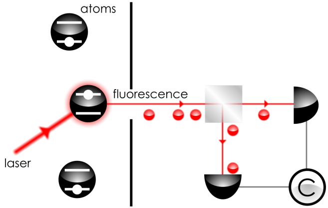

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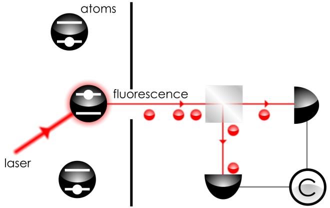

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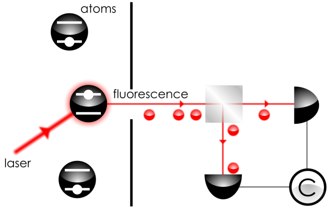

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Classical probabilities

- Stochastic variable X, values x (continuous)
- Probability density: $p(x) = \langle \delta(x - X) \rangle$
- Mean values (moments):

$$\langle X^n \rangle = \int dx\, x^n p(x) = \int dx\, x^n \langle \delta(x - X) \rangle$$

- Using Fourier representation, $\delta(x - X) = \frac{1}{2\pi} \int dk\, e^{ik(x-X)}$:

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Taylor expansion of characteristic function:

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Phase-space distributions

- Basic operators $\hat{x}$, $\hat{p}$: $[\hat{x}, \hat{p}] \neq 0$ (non-commuting)
- Phase-space distribution: ambiguous ordering of $\hat{x}$, $\hat{p}$

$$W(x,p) = \frac{1}{(2\pi)^2} \int dk \int dk' \, e^{-i(kx-k'p)} \langle e^{i(k\hat{x}-k'\hat{p})} \rangle$$

- Alternative representation (oscillator, radiation mode): $\hat{x}, \hat{p} \rightarrow \hat{a}, \hat{a}^\dagger$

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- Normal ordering: Glauber-Sudarshan P function $\Rightarrow$ singular, $P(\alpha) \not\geq 0$

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s-parametrized quasiprobabilities

- Relation between representations:

$$P(\alpha; s) = \frac{1}{\pi^2} \int d^2\beta \, e^{\alpha\beta^* - \alpha^*\beta} \langle \hat{D}(\beta; s) \rangle, \quad \hat{D}(\beta; s) = e^{\frac{s}{2}|\beta|^2} \hat{D}(\beta)$$

- special cases:

$$s = 1 \to P(\alpha), \quad s = 0 \to W(\alpha), \quad s = -1 \to Q(\alpha)$$

- Convolution of P function with Gaussian noise (for $s < 1$):

$$P(\alpha; s) = \frac{2}{\pi(1-s)} \int d^2\beta \exp\left(-\frac{2|\alpha - \beta|^2}{(1-s)}\right) P(\beta)$$

- Increasing noise with decreasing $s \Rightarrow$ better experimental handling!

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$$P(\alpha; s) = \frac{1}{\pi^2} \int d^2\beta \, e^{\alpha\beta^* - \alpha^*\beta} \langle \hat{D}(\beta; s) \rangle, \quad \hat{D}(\beta; s) = e^{\frac{s}{2}|\beta|^2} \hat{D}(\beta)$$

- special cases:

$$s = 1 \to P(\alpha), \quad s = 0 \to W(\alpha), \quad s = -1 \to Q(\alpha)$$

- Convolution of P function with Gaussian noise (for $s < 1$):

$$P(\alpha; s) = \frac{2}{\pi(1-s)} \int d^2\beta \exp\left(-\frac{2|\alpha - \beta|^2}{(1-s)} \right) P(\beta)$$

- Increasing noise with decreasing $s \Rightarrow$ better experimental handling!

Nonclassical states

- P representation of a general quantum state:[4] $\hat{\rho} = \int d^2\alpha P(\alpha)|\alpha\rangle\langle\alpha|$
- P contains full information on the quantum state
- Properties:
  - Normalization: $\int d^2\alpha P(\alpha) = 1$
  - $P(\alpha) \neq P_{\rm cl}(\alpha)$: $P(\alpha) \ngeq 0$ and strongly singular
- State nonclassical: $P(\alpha) < 0$
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Experimental characterization of P functions

- Characteristic functions of quadratures, $x_\varphi = \hat{a}e^{i\varphi} + \hat{a}^\dagger e^{-i\varphi}$:

$$G(k,\varphi) = \left\langle e^{ik\hat{x}_\varphi} \right\rangle = \int dx\, p(x,\varphi) e^{ikx}$$

- Sampling of characteristic function $\langle : \hat{D}(\beta) : \rangle$:

$$\Phi(ike^{-i\varphi}) \approx e^{\frac{1}{2}k^2} \frac{1}{N} \sum_{j=1}^{N} e^{ikx_\varphi(j)}$$

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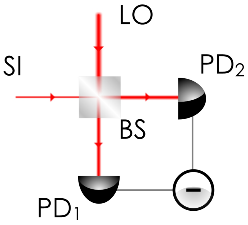

Balanced homodyne detection

Nonclassical characteristic functions

- Characteristic function $\Phi(\beta)$ of $P(\alpha)$: always regular
- Theorem (Bochner 1933):[5]
  $P(\alpha)$ probability iff for any smooth function $f(\alpha)$ with compact support

$$\iint d^2\alpha\, d^2\beta\; \Phi(\alpha-\beta)\, f^*(\alpha) f(\beta) \geqslant 0$$

- Discrete version: $\sum_{i,j=1}^{n} \Phi(\beta_i-\beta_j)\,\xi_i^*\,\xi_j \geq 0,$

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Nonclassicality criteria

- Define matrix: $\Phi_{ij} = \Phi(\beta_i - \beta_j)$
- **Theorem:** A continuous function $\Phi(\beta)$ with $\Phi(0) = 1$ and $\Phi^*(\beta) = \Phi(-\beta)$ is a classical characteristic function, iff for any k

$$D_k \equiv D_k(\beta_1, \ldots \beta_k) = \begin{vmatrix} 1 & \Phi_{12} & \cdots & \Phi_{1k} \\ \Phi_{12}^* & 1 & \cdots & \Phi_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ \Phi_{1k}^* & \Phi_{2k}^* & \cdots & 1 \end{vmatrix} \geq 0$$

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Nonclassicality of first order

- Observable characteristic functions of quadratures:

$$G(k,\varphi) = G_{\rm gr}(k)\,\Phi(ike^{-i\varphi}) \quad \Leftrightarrow \quad \mathrm{FT}[p(x,\varphi)]$$

- Ground (vacuum) state:

$$\Phi_{\rm gr} = 1 \quad \Leftrightarrow \quad G_{\rm gr}(k) = \exp\left(-\frac{k^2}{2}\right)$$

- First-order nonclassicality:[7]

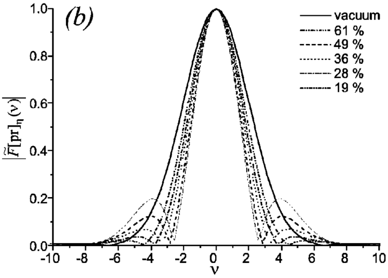
$$D_2 < 0: \quad |\Phi(\beta)| > 1 \quad \Leftrightarrow \quad |G(k,\varphi)| > G_{\rm gr}(k)$$

$\Rightarrow$ Necessary and sufficient for pure quantum states

[7] W. Vogel, Phys. Rev. Lett. **84**, 1849 (2000)

Nonclassicality of first order

- Experiment:[8] $\hat{\rho} = \eta|1\rangle\langle 1| + (1-\eta)|0\rangle\langle 0|$



[8] A.I. Lvovsky and J.H. Shapiro, Phys. Rev. A **65**, 033830 (2002)

Nonclassicality of higher order

- Experiment:[9]

$$\hat{\rho} = \mathcal{N}\hat{a}^\dagger\hat{\rho}_{\rm th}\hat{a}$$

SPATS (Single Photon Added Thermal State)

- Nonclassicality of first (a) and second order (b): $D_3 < 0$

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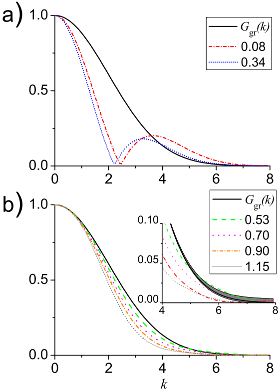

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Phase-diffused squeezed state

- Experiment:[10] phase-diffused squeezed vacuum
- Wigner function:

$$W(\alpha) = \int f(\varphi) \frac{1}{2\pi\sqrt{V_x V_p}} \exp\left\{-\frac{\mathrm{Re}^2(\alpha e^{-i\varphi})}{2V_x} - \frac{\mathrm{Im}^2(\alpha e^{-i\varphi})}{2V_p}\right\} d\varphi,$$

with $V_x = 0.36$, $V_p = 5.28$

- Gaussian phase distribution $f(\varphi)$ with variance σ^2
- Squeezing for $\sigma < 22.2°$

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Nonclassicality with phase noise

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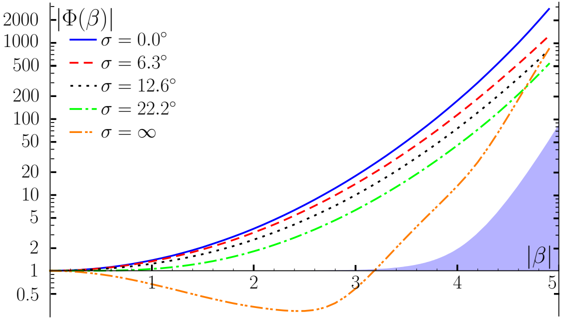

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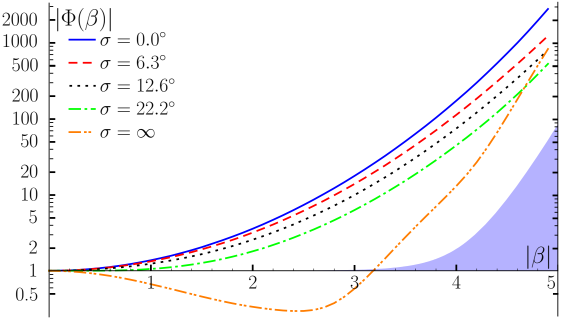

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Characterization of nonclassicality by moments

- Nonclassicality: P-function is not a probability distribution,

$$P(\alpha) \neq P_{cl}(\alpha)$$

- Equivalent condition: $\exists \hat{f} : \quad \langle : \hat{f}^{\dagger}\hat{f} : \rangle < 0,$

  where $\langle : \hat{f}^{\dagger}\hat{f} : \rangle = \int dP(\alpha)|f(\alpha)|^2$

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$$\hat{f} = \int d^2\alpha \, \underline{f}(\alpha) : \hat{D}(-\alpha) :$$

$\Rightarrow$ Bochner condition!

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- Quadrature expansion:[11] $\hat{f} = f(\hat{x}_\varphi, \hat{p}_\varphi) = \sum_{n,m} f_{nm} : \hat{x}_\varphi^n \hat{p}_\varphi^m :$
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[11] E. Shchukin, Th. Richter, and W. Vogel, Phys. Rev. A **71**, 011802(R) (2005)
[12] G.S. Agarwal, Opt. Comm. **95**, 109 (1993)

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Moments of annihilation and creation operators

- Quadratic form:[13] $\langle : \hat{f}^\dagger \hat{f} : \rangle = \sum_{n,m,k,l} c^*_{nm} c_{kl} \langle \hat{a}^{\dagger m+k} \hat{a}^{n+l} \rangle$
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Nonlinear amplitude squeezing

- Higher-order amplitude squeezing:[14]

$$\Delta^{(k)} = \begin{vmatrix} 1 & \langle \hat{a}^{\dagger k} \rangle & \langle \hat{a}^{k} \rangle \\ \langle \hat{a}^{k} \rangle & \langle \hat{a}^{\dagger k}\hat{a}^{k} \rangle & \langle \hat{a}^{2k} \rangle \\ \langle \hat{a}^{\dagger k} \rangle & \langle \hat{a}^{\dagger 2k} \rangle & \langle \hat{a}^{\dagger k}\hat{a}^{k} \rangle \end{vmatrix} < 0,$$

$$\Delta^{(k)} = \frac{1}{4}\min_{\varphi}\left\langle : \left(\Delta\hat{x}_{\varphi}^{(k)}\right)^{2} : \right\rangle \max_{\varphi}\left\langle : \left(\Delta\hat{x}_{\varphi}^{(k)}\right)^{2} : \right\rangle,$$

where $\hat{x}_{\varphi}^{(k)} = \hat{a}^{k} e^{i\varphi} + \hat{a}^{\dagger k} e^{-i\varphi}$

- Amplitude-squared squeezing:[15] $k = 2$

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[14] E. Shchukin and W. Vogel, J. Phys: C.S. **36**, 183 (2006)

[15] M. Hillery, Phys. Rev. A **72**, 3796 (1987)

Nonlinear amplitude squeezing

- Higher-order amplitude squeezing:[14]

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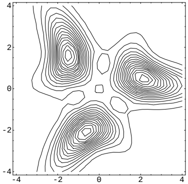
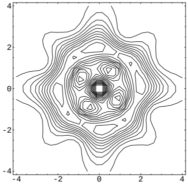




[14] E. Shchukin and W. Vogel, J. Phys: C.S. **36**, 183 (2006)
[15] M. Hillery, Phys. Rev. A **72**, 3796 (1987)

Experimental characterization by moments

- Phase-diffused squeezed vacuum[16]
- Normally ordered quadrature moments:[17] $\langle : \hat{x}^k : \rangle \rightarrow$ test up to $k = 14$

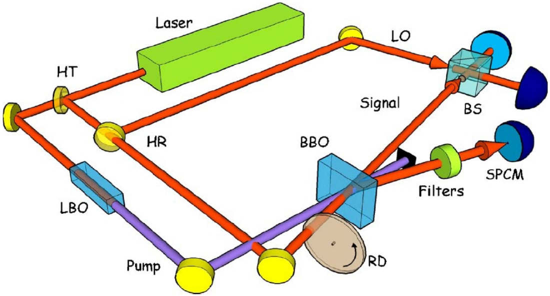
$\Rightarrow$ Mininal eigenvalues of matrices

$\sigma/^\circ$	2×2 Matrix	4×4 Matrix	6×6 Matrix	8×8 Matrix
0.0	$-0.6362(1 \pm 0.25\%)$	$-4.294(1 \pm 0.86\%)$	$-104.0(1 \pm 2.5\%)$	$-6201(1 \pm 6.1\%)$
6.3	$-0.5717(1 \pm 0.03\%)$	$-3.337(1 \pm 0.11\%)$	$-69.93(1 \pm 0.35\%)$	$-3593(1 \pm 0.98\%)$
12.6	$-0.4060(1 \pm 0.08\%)$	$-2.040(1 \pm 1.1\%)$	$-6.728(1 \pm 53\%)$	$-107.4(1 \pm 110\%)$
22.2	$0.0197(1 \pm 3.0\%)$	$-0.2323(1 \pm 1.1\%)$	$-0.5358(1 \pm 4.1\%)$	$-2.299(1 \pm 71\%)$
∞	$1.0000(1 \pm 0\%)$	$0.7856(1 \pm 1.2\%)$	$0.5493(1 \pm 12\%)$	$10.85(1 \pm 13\%)$

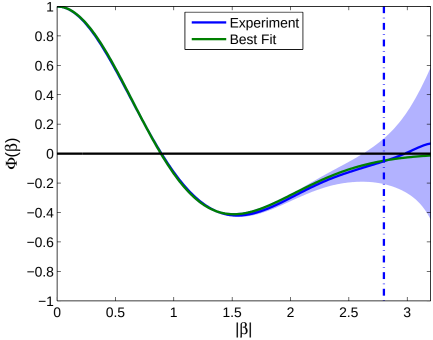
[16] Kiesel, Vogel, Hage, DiGuglielmo, Samblowski, Schnabel, Phys. Rev. A **79**, 022122 (2009)
[17] G.S. Agarwal, Opt. Comm. **95**, 109 (1993)

Regular nonclassical *P* functions

- Single photon: $P(\alpha) = \left(1 + \frac{\partial}{\partial\alpha}\frac{\partial}{\partial\alpha^*}\right)\delta(\alpha)$
- SPATS: $\delta(\alpha) \Rightarrow$ Gaussian distribution $\Rightarrow P(\alpha)$ regular



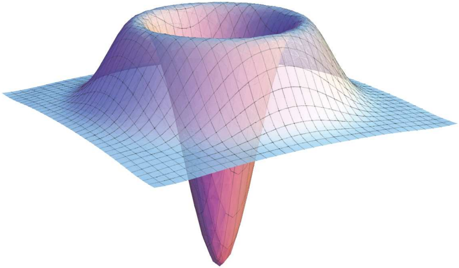
Characteristic function of SPATS



SPATS: $n_{\rm th} \approx 1.1$ and $\eta = 0.6$

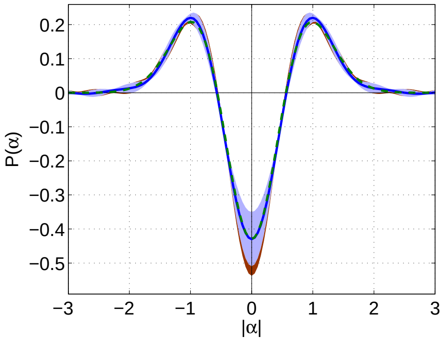
P function of SPATS

Via Hankel transform:[18] $$P(\alpha) = \frac{2}{\pi}\int_0^{|\beta|c} bJ_0(2b|\alpha|)\Phi(b)db$$



[18] Kiesel, Vogel, Zavatta, Parigi, Bellini, Phys. Rev. A **78**, 021804(R) (2008)

Experimental noise



Statistical significance of negativity: 5 standard deviations

Nonclassicality filter

- Problem: P is singular $\Leftrightarrow$ Φ is not integrable
- Solution: filter characteristic function: $\Phi_\Omega(\beta) = \Phi(\beta)\Omega_w(\beta)$
- Requirements for the filter $\Omega_w(\beta)$ with width w:[19]
  - $\Phi_\Omega(\beta)$ is integrable $\Rightarrow$ regularized function $P_\Omega(\alpha)$
  - $P_{\Omega,\mathrm{cl}}(\alpha) \geq 0 \Rightarrow$ Fourier transform of $\Omega_w(\beta)$ is nonnegative
  - Original quantum state: $\lim_{w\to\infty} \Omega_w(\beta) = 1$; $\quad P_\Omega \Rightarrow P$
  - Suppression of statistical errors: $\Omega_w(\beta)e^{|\beta|^2/2}$ to be integrable
- **Such filters exist for arbitrary quantum states!**

[19] T. Kiesel and W. Vogel, PRA **82**, 032107 (2010)

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Nonclassicality quasiprobabilities

- Additional requirement: $\Omega_w(\beta) \neq 0 \;\Rightarrow$ Filter is invertible
- Full information on the quantum state[20]
- Regularized $P_\Omega \Rightarrow$ *nonclassicality quasi-probability:* $P_{\rm Ncl}$
- Construction of a nonclassicality filter:
  - Rapidly decaying function: $\omega(\beta) = e^{-|\beta|^4}$
  - Fourier transform has negativities
  - Autocorrelation function: $\Omega_1(\beta) \sim \int \omega(\beta')\omega(\beta+\beta')d^2\beta'$
  - Fourier transform of $\Omega_1(\beta)$ is nonnegative

[20] G.S. Agarwal and E. Wolf, PRD **2**, 2161 (1970)

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For any nonclassical quantum state one finds negativities in $P_\Omega(\alpha)$ for sufficiently large filter width w.

Scheme for a universal nonclassicality test

1. Reconstruct characteristic function $\Phi(\beta)$ from balanced homodyne detection
2. Choose filter $\Omega_w(\beta)$ and calculate $\Phi_\Omega(\beta) = \Phi(\beta)\Omega_w(\beta)$
3. Calculate nonclassicality quasiprobability $P_{\rm Ncl}(\alpha)$ by Fourier transform
4. Increase filter width until negativities appear

- Method is only limited by statistical uncertainties

$\Rightarrow$ Increasing the width w: larger negativities, but larger statistical noise

Related phase-space methods

- Filtered characteristic function: $\Phi_\Omega(\beta) = \Phi(\beta)\Omega(\beta)$
- Klauder's regularization: infinitely differentiable filtered *P* function[21]
  - Not aimed at nonclassicality filtering
- *s*-parameterized quasi-probabilities[22]
  - Filter $\Omega_s(\beta) = e^{(s-1)|\beta|^2/2}$
  - Fails for displaying nonclassicality by negativities, e.g. for squeezed states
- Filter $\Omega(\beta)$ for generalized phase-space functions and operator ordering[23]
  - Additional constraints needed for detection of nonclassicality

[21] J.R. Klauder, PRL **16**, 534 (1966)
[22] K.E. Cahill and R.J. Glauber, PR **177**, 1882 (1969)
[23] G.S. Agarwal and E. Wolf, PRD **2**, 2161 (1970)

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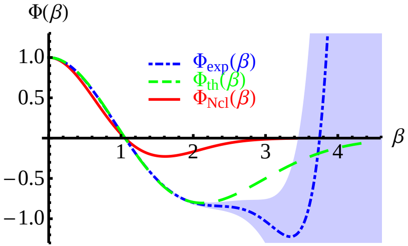
P_{Ncl} of SPATS

Single-photon-added thermal state[24] ($\bar{n}_{\mathrm{th}} \approx 0.5$, $w = 1.4$)



[24] T. Kiesel, W. Vogel, M. Bellini, A. Zavatta, PRA **83**, 032116 (2011)

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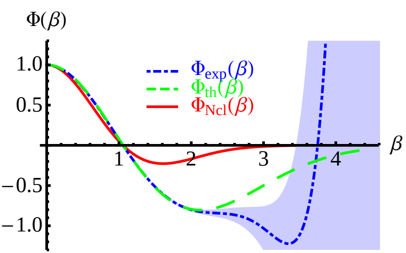
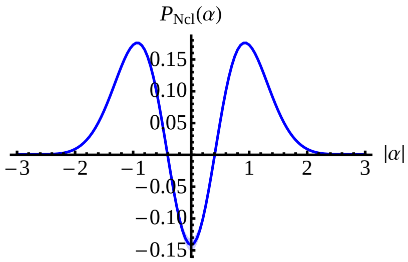




Statistical significance of $P_{\mathrm{Ncl}}(\alpha) < 0$: 15 standard deviations

[24] T. Kiesel, W. Vogel, M. Bellini, A. Zavatta, PRA **83**, 032116 (2011)

P_{Ncl} of squeezed vacuum

- P function of squeezed vacuum $\Rightarrow$ demanding regularization:

$$P_{\mathrm{sv}}(\alpha) = e^{-\frac{V_x - V_p}{8}\left(\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\alpha^{*2}} - 2\frac{V_x+V_p-2}{V_x-V_p}\frac{\partial}{\partial\alpha}\frac{\partial}{\partial\alpha^*}\right)}\delta(\alpha)$$

- Direct sampling of P_{Ncl}:[25] $P_\Omega(\alpha) \approx \frac{1}{N}\sum_{i=1}^{N} f_\Omega(x_i, \varphi_i; \alpha, w)$

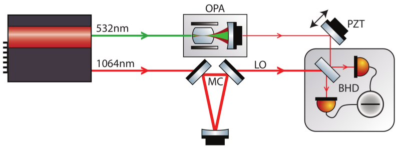
- Pattern function:

$$f_\Omega(x, \varphi; \alpha, w) = \int_{-\infty}^{\infty} db\, \frac{|b|}{\pi} e^{ibx} e^{2i|\alpha| b \sin(\arg(\alpha) - \varphi - \frac{\pi}{2})} e^{b^2/2} \Omega_w(b)$$

[25] T. Kiesel, W. Vogel, B. Hage, R. Schnabel, Phys. Rev. Lett., in press; arXiv:1105.4591

P_{Ncl} of squeezed vacuum

Experimental setup and optimal filtering



P_{Ncl} of squeezed vacuum

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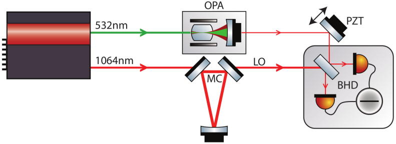
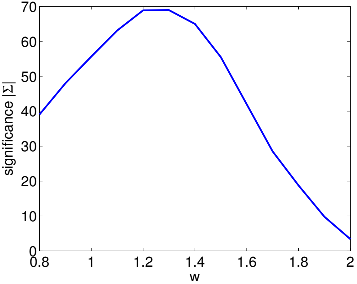
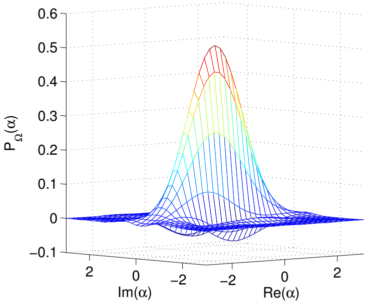




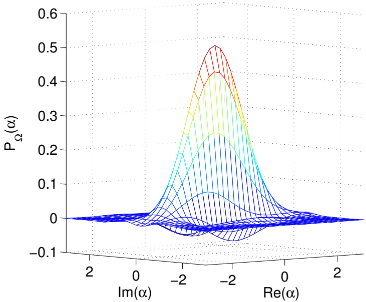
Statistical significance of negativity of $P_{\mathrm{Ncl}}(\alpha)$: max. 69 standard deviations

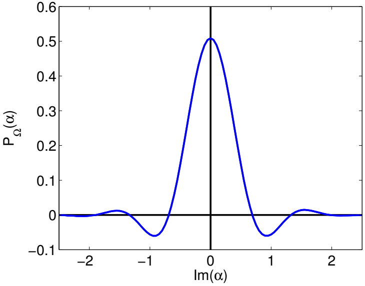
P_{Ncl} of squeezed vacuum



Squeezed state, optimal cutoff: $w = 1.3$

P_{Ncl} of squeezed vacuum





Squeezed state, optimal cutoff: $w = 1.3$

Summary: Nonclassical *P* functions

- *P* representation: $\hat{\rho} = \int dP(\alpha)|\alpha\rangle\langle\alpha|$
- Quasiprobability: $P(\alpha) \ngeq 0$, singular
- Direct sampling of characteristic function:

  $\Phi(ike^{-i\varphi}) \approx e^{\frac{1}{2}k^2} \frac{1}{N} \sum_{j=1}^{N} e^{ikx_\varphi(j)}$
- Nonclassicality $\Leftarrow |\Phi(\beta)| > 1$
- Regularization: $P(\alpha) \Rightarrow P_{\mathrm{Ncl}}(\alpha)$
  - Applies to all quantum states
  - Suppression of experimental noise

Summary: Nonclassical *P* functions

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- Quasiprobability: $P(\alpha) \not\geq 0$, singular
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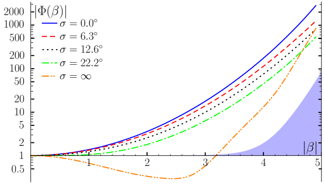

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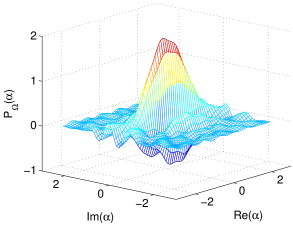

Squeezed state, cutoff: $w = 1.8$