Reconstruction of Quasi-Probability Distributions

II. Quantum Correlation Effects

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Contents

Definition of entanglement

- A bipartite state $\hat{\sigma}$ is called separable if it is a convex combination of factorizable ones:[1]

$$\hat{\sigma} = \sum_{a,b} p_{\rm cl}(a,b)|a,b\rangle\langle a,b|,$$

where $|a,b\rangle = |a\rangle \otimes |b\rangle$, $p_{\rm cl}(a,b) \geqslant 0$, and $\sum_{a,b} p_{\rm cl}(a,b) = 1$.

- **Def.:** State $\hat{\varrho}$ is entangled if it cannot be given in such a form: $\hat{\varrho} \neq \hat{\sigma}$
- Separable states $\Rightarrow$ classical correlations:

$$\langle \hat{A} \otimes \hat{B} \rangle = \sum_{a,b} p_{\rm cl}(a,b)\langle a|\hat{A}|a\rangle\langle b|\hat{B}|b\rangle$$

[1] R. F. Werner, Phys. Rev. A, **40**, 4277 (1989)

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Relevance of entanglement

- Early studies: EPR paradoxon[2] and Schrödinger's cat.[3]
- Nowadays: entanglement as key resource for[4]
  - Quantum information processing
  - Quantum computation
  - Quantum technology

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[2] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).
[3] E. Schrödinger, Naturwiss. **23**, 807 (1935).
[4] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000);
R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009);
O. Gühne and G. Tóth, Physics Reports **474**, 1 (2009).

Simple situations

- Separable pure state:

$$|\psi\rangle = \frac{1}{2}\left(|0,0\rangle + |1,0\rangle + |0,1\rangle + |1,1\rangle\right) = \frac{1}{2}\left(|0\rangle + |1\rangle\right) \otimes \left(|0\rangle + |1\rangle\right)$$

- Entangled pure state:

$$|\psi\rangle = \frac{1}{2}\left(|0,0\rangle + |1,0\rangle + |0,1\rangle - |1,1\rangle\right) \neq |a\rangle \otimes |b\rangle$$

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Entanglement witnesses

Definition:[5]

An entanglement witness $\hat{W}$ is a bounded Hermitian operator with

$$\mathrm{tr}(\hat{\sigma}\hat{W}) \geq 0 \; (\forall \hat{\sigma} \; \mathrm{separable}), \quad \exists \hat{\varrho} : \mathrm{tr}(\hat{\varrho}\hat{W}) < 0$$

$\Rightarrow$ General test, if general form of $\hat{W}$ is known!

[5] M. Horodecki, P. Horodecki and R. Horodecki, Phys. Lett. A **223**, 1 (1996).

Entanglement witnesses

Lemma:[5] For any entanglement witness $\hat{W}$ exists a real number $\lambda > 0$ and a positive Hermitian operator $\hat{C}$ so that

$$\hat{W} = \lambda \hat{1} - \hat{C}.$$

Proof:

$$\hat{W} = \sum_i w_i |w_i\rangle\langle w_i| = \lambda \sum_i |w_i\rangle\langle w_i| - \sum_i \underbrace{(\lambda - w_i)}_{\geq 0} |w_i\rangle\langle w_i| \equiv \lambda \hat{1} - \hat{C}$$

where $\lambda \geq w_{\rm max}$, $w_{\rm max}$ being the largest eigenvalue of $\hat{W}$

[5] J. Sperling and W. Vogel, Phys. Rev. A **79**, 022318 (2009)

General optimized entanglement conditions

- For separable states σ:[6]
  - $\hat{L}$ Hermitian operator:

    $\mathrm{Tr}(\hat{\sigma}\hat{L}) \leq f_{AB}(\hat{L})$
  - $f_{AB}(\hat{L}) = \max\{\langle a, b|\hat{L}|a, b\rangle\}$
- Optimized necessary and sufficient entanglement conditions:
- Optimal test: identifies maximal number of entangled states accessible by $\hat{L}$!

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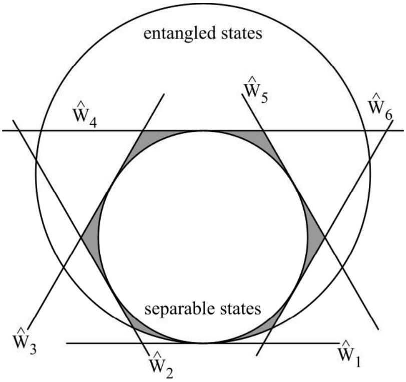
Optimized entanglement witnesses



Optimization: separability eigenvalue problem

- $f_{AB}(\hat{L})$: extremum of $g(a,b) = \langle a, b|\hat{L}|a, b\rangle$,
- Normalization conditons: $\langle a|a\rangle = 1$, $\langle b|b\rangle = 1$
- Lagrange multipliers: separability eigenvalue (SE) equations:[7]
  - $\hat{L}_b = \mathrm{Tr}_B(\hat{L}[\mathbb{I} \otimes |b\rangle\langle b|])$ and $\hat{L}_a = \mathrm{Tr}_A(\hat{L}[|a\rangle\langle a| \otimes \mathbb{I}])$

$\Rightarrow$ Solution: $f_{AB}(\hat{A}) = \sup\{g\}$

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Example: phase randomized squeezed-vacuum

- Two-mode squeezed-vacuum state:

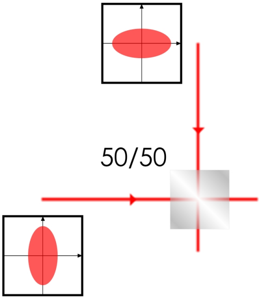
$$|\xi\rangle = \sqrt{1-\xi^2}\sum_{k=0}^{\infty}\xi^k|k,k\rangle$$

- Phase randomization:

$$\rho_{\delta\varphi} = \frac{1}{2\delta\varphi}\int_{-\delta\varphi}^{+\delta\varphi} d\varphi\,|\xi e^{i\varphi}\rangle\langle\xi e^{i\varphi}|$$

- Full randomization $\delta\varphi = \pi$:

$$\rho_\pi = (1-\xi^2)\sum_{k=0}^{\infty}\xi^{2k}|k,k\rangle\langle k,k|$$



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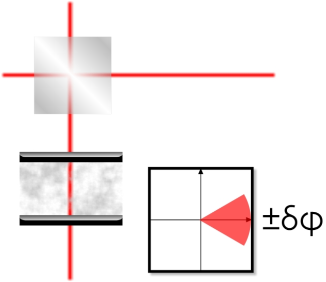

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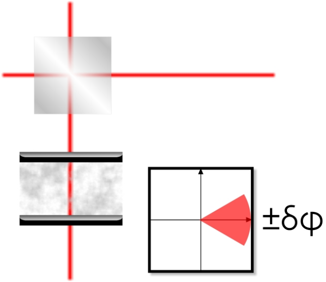

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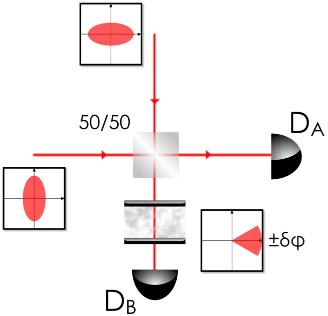

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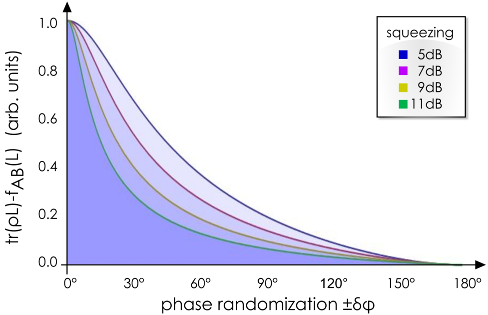

Figure: Optimized entanglement condition.

- $\hat{L} = \sum_{k,l} \mathrm{sinc}(\delta\varphi[k-l])|k,k\rangle\langle l,l|,$ with $f_{AB}(\hat{L}) = 1$

Entanglement in terms of uncorrelated states

- $\hat{\rho}$ in terms of product states:[9]

$$\hat{\rho} = (1+\mu)\hat{\sigma} - \mu\hat{\sigma}' \; (\mu \geq 0) \; \Leftrightarrow \; \hat{\rho} = \int dP(a,b) |a,b\rangle\langle a,b|$$

$\Rightarrow$ $P(a,b) \neq P_{\mathrm{cl}}(a,b)$; possible conclusions:

- $P(a,b) \geq 0 \quad \Rightarrow \quad$ quantum state is separable
- $P(a,b) < 0 \quad \not\Rightarrow \quad$ entangled

- Problem: $P(a,b)$ is ambiguous
- Optimization needed: $P_{\mathrm{Ent}}(a,b)$ with minimal negativity

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Construction / reconstruction of quasi-probabilities

- Spectral decomposition of state $\hat{\rho}$: $\hat{\rho} = \sum_i p_{\phi_i} |\phi_i\rangle\langle\phi_i|$
- Schmidt decomposition of all states $|\phi_i\rangle$: $|\phi\rangle = \sum_{k=1}^{r(\phi)} \lambda_k |e_k, f_k\rangle$

$$|\phi\rangle\langle\phi| = \sum_k \lambda_k^2 |e_k, f_k\rangle\langle e_k, f_k| + \sum_{k>l} \lambda_k \lambda_l \left(|e_k, f_k\rangle\langle e_l, f_l| + |e_l, f_l\rangle\langle e_k, f_k| \right)$$

- Representation with factorized states:[10]

$$|e_k, f_k\rangle\langle e_l, f_l| + |e_l, f_l\rangle\langle e_k, f_k| = \sum_{n=0}^{3} (-1)^n |\mathfrak{s}_n^{(k,l)}, \mathfrak{s}_n^{(k,l)}\rangle\langle \mathfrak{s}_n^{(k,l)}, \mathfrak{s}_n^{(k,l)}|$$

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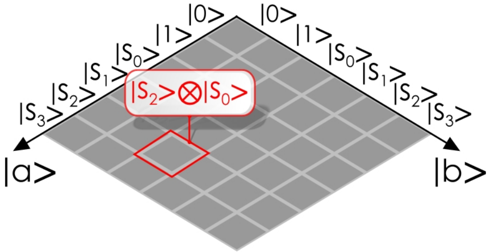
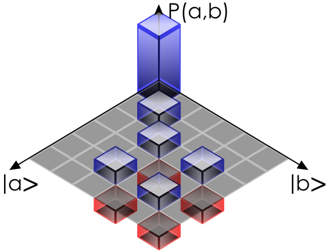
Visualisation of Quasi-Probabilities



Local superposition states: $|\mathfrak{s}_n\rangle = \frac{1}{\sqrt{1+\xi}}(|0\rangle + i^n\sqrt{\xi}|1\rangle)$

Ambiguous quasiprobability representation

phase-randomized two-mode squeezed vacuum (5 dB squeezing)



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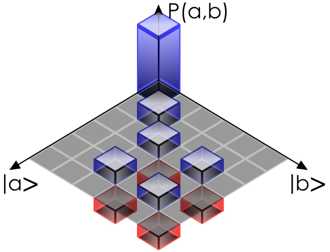

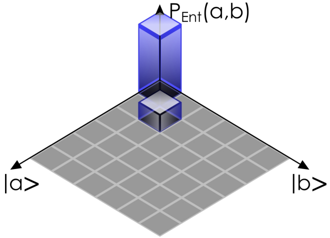

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Towards unambiguous quasiprobabilities

- Signed measures f generating the $\hat{0}$ operator:

$$\hat{0} = \int df(a,b)|a,b\rangle\langle a,b| \quad \Rightarrow \quad \hat{\rho} \equiv \hat{\rho} + \hat{0}$$

- New quasiprobabilities: $P_{\rm Ent}(a,b) \equiv P(a,b) + f(a,b)$, with

$$P_{\rm Ent}: \quad \int d|P(a,b) + f(a,b)|^2 \to \min$$

- Optimization with respect to separability norm:

$$||\hat{\rho} - \hat{\sigma}||_{\rm sep} = \sup\{|\langle a,b|\hat{\rho} - \hat{\sigma}|a,b\rangle|\} \to \min$$

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Entanglement quasiprobabilities P_{Ent}

- Optimization procedure:[11]

  (1) Separability eigenvalue (SE) problem:

  $$\hat{\rho}_{a_i}|b_i\rangle = g_i|b_i\rangle, \quad \hat{\rho}_{b_i}|a_i\rangle = g_i|a_i\rangle; \quad \text{with} \quad \hat{\rho}_{a_i} = \langle a_i|\,\hat{\rho}\,|a_i\rangle$$

  (2) State as combination of SE vectors: $\hat{\rho} = \sum_i p_i |a_i, b_i\rangle\langle a_i, b_i|$

  (3) Linear system for p_i: $\langle a_j, b_j|\hat{\rho}|a_j, b_j\rangle = g_j$

  (4) Solution: $p_i = p(a_i, b_i) \Rightarrow P_{\mathrm{Ent}}(a, b)$

$\hat{\rho}$ **entangled** $\Leftrightarrow$ $\exists a, b$ **with** $P_{\mathrm{Ent}}(a, b) < 0$

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$\hat{\rho}$ **entangled** $\Leftrightarrow$ $\exists a, b$ **with** $P_{\mathrm{Ent}}(a, b) < 0$

[11] J. Sperling and W. Vogel, PRA **79**, 042337 (2009)

Entanglement quasiprobabilities P_{Ent}

- Optimization procedure:[11]

(1) Separability eigenvalue (SE) problem:

$$\hat{\rho}_{a_i}|b_i\rangle = g_i|b_i\rangle, \quad \hat{\rho}_{b_i}|a_i\rangle = g_i|a_i\rangle; \quad \text{with} \quad \hat{\rho}_{a_i} = \langle a_i|\,\hat{\rho}\,|a_i\rangle$$

(2) State as combination of SE vectors: $\hat{\rho} = \sum_i p_i |a_i, b_i\rangle\langle a_i, b_i|$

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Entanglement quasiprobabilities P_{Ent}

- Two-mode squeezed-vacuum: optimized P_{Ent} representation

$$\rho_{\delta\varphi} = \frac{1}{1+\xi^2}|0,0\rangle\langle 0,0| + \frac{\xi^2}{1+\xi^2}|1,1\rangle\langle 1,1| + \frac{(1+\xi)^2\mathrm{sinc}(\delta\varphi)}{8(1+\xi^2)}(\dots)$$

$$\dots = +|\mathfrak{s}_0,\mathfrak{s}_0\rangle\langle\mathfrak{s}_0,\mathfrak{s}_0| + |\mathfrak{s}_1,\mathfrak{s}_3\rangle\langle\mathfrak{s}_1,\mathfrak{s}_3| + |\mathfrak{s}_2,\mathfrak{s}_2\rangle\langle\mathfrak{s}_2,\mathfrak{s}_2| + |\mathfrak{s}_3,\mathfrak{s}_1\rangle\langle\mathfrak{s}_3,\mathfrak{s}_1| - |\mathfrak{s}_0,\mathfrak{s}_2\rangle\langle\mathfrak{s}_0,\mathfrak{s}_2| - |\mathfrak{s}_1,\mathfrak{s}_1\rangle\langle\mathfrak{s}_1,\mathfrak{s}_1| - |\mathfrak{s}_2,\mathfrak{s}_0\rangle\langle\mathfrak{s}_2,\mathfrak{s}_0| - |\mathfrak{s}_3,\mathfrak{s}_3\rangle\langle\mathfrak{s}_3,\mathfrak{s}_3|$$

- Local superposition states:

$$|\mathfrak{s}_n\rangle = \frac{1}{\sqrt{1+\xi}}\left(|0\rangle + i^n\sqrt{\xi}|1\rangle\right)$$

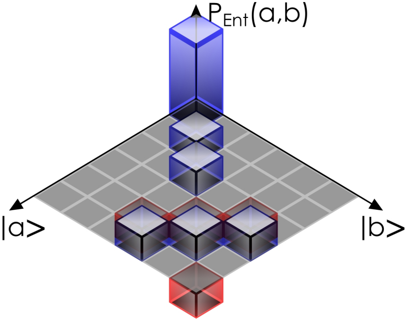

Figure: $\xi = 0.5$, $\delta\varphi = 0^\circ$.

Entanglement quasiprobabilities P_{Ent}

- Two-mode squeezed-vacuum: optimized P_{Ent} representation

$$\rho_{\delta\varphi} = \frac{1}{1+\xi^2}|0,0\rangle\langle 0,0| + \frac{\xi^2}{1+\xi^2}|1,1\rangle\langle 1,1| + \frac{(1+\xi)^2 \mathrm{sinc}(\delta\varphi)}{8(1+\xi^2)}(\ldots)$$

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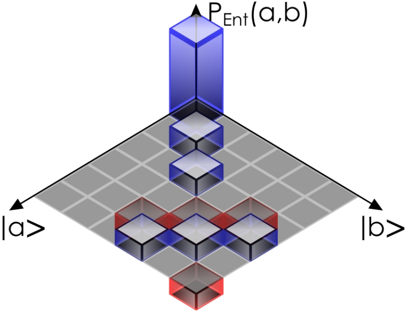

Figure: $\xi = 0.5$, $\delta\varphi = 90^\circ$.

Entanglement quasiprobabilities P_{Ent}

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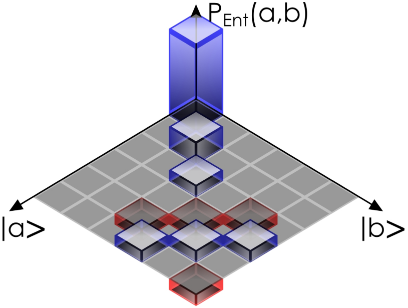

Figure: $\delta\varphi = 120°$.

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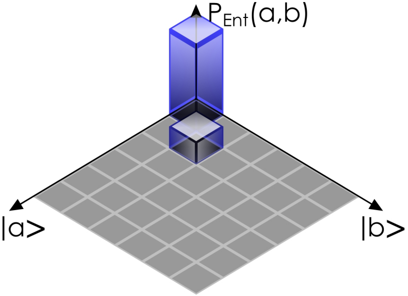
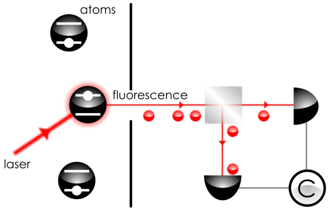


Figure: $\xi = 0.5$, $\delta\varphi = 180^\circ$.

Photon antibunching

- First demonstration of nonclassical light: photon antibunching[12]



[12]H.J. Kimble, M. Dagenais, and L. Mandel, PRL **39**, 691 (1977)

Photon antibunching

- Violation of Schwarz inequality: $\langle \mathcal{T} {:} \hat{I}(0)\hat{I}(\tau) {:} \rangle > \langle {:} [\hat{I}(0)]^2 {:} \rangle$

VOLUME 39, NUMBER 11 PHYSICAL REVIEW LETTERS 12 SEPTEMBER 1977 691

Photon Antibunching in Resonance Fluorescence

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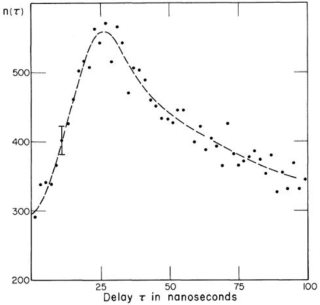
Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627
(Received 22 July 1977)

The phenomenon of antibunching of photoelectric counts has been observed in resonance fluorescence experiments in which sodium atoms are continuously excited by a dye-laser beam. It is pointed out that, unlike photoelectric bunching, which can be given a semiclassical interpretation, antibunching is understandable only in terms of a quantized electromagnetic field. The measurement also provides rather direct evidence for an atom undergoing a quantum jump.

$$\langle n(\tau)\rangle = N_s \Delta\tau \alpha_2 \langle \mathcal{T}{:}\hat{I}_1(t)\hat{I}_2(t+\tau){:}\rangle / \langle \hat{I}_1(t)\rangle, \qquad (6)$$



General nonclassical field correlations

- P function $\Rightarrow$ P functional:[13]

$$P(\{E^{(+)}(i)\}) = \left\langle \mathcal{T} \colon \prod_{i=1}^{k} \hat{\delta}\left(\hat{E}^{(+)}(i) - E^{(+)}(i)\right) \colon \right\rangle, \quad i \equiv (\mathbf{r}_i, t_i)$$

- Nonclassical correlations: $P \not\geq 0 \quad \Leftrightarrow \quad \exists \hat{f} \colon \langle \mathcal{T} \colon \hat{f}^\dagger \hat{f} \colon \rangle < 0$

$\Rightarrow$ Hierarchy of conditions for field correlation functions, such as:

$$|\langle \mathcal{T} \colon \hat{E}(1)\hat{I}(2) \colon \rangle| > \sqrt{\langle \colon [\hat{E}(1)]^2 \colon \rangle \langle \colon [\hat{I}(2)]^2 \colon \rangle}$$

- Detection: balanced homodyne correlation measurement[14]

[13] W. Vogel, PRL **100**, 013605 (2008)
[14] E. Shchukin and W. Vogel, PRL **96**, 200403 (2006)

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Summary and Conclusions

Entanglement:

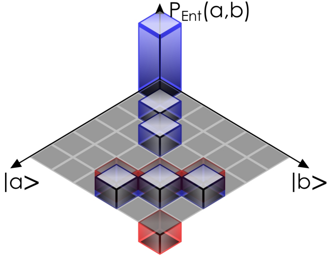
- Representation of entangled states:

$$\hat{\rho} = \int dP(a,b)|a,b\rangle\langle a,b|$$

- Optimization: SE problem of $\hat{\rho}$

$$P(a,b) \Rightarrow P_{\mathrm{Ent}}(a,b)$$

- $P_{\mathrm{Ent}}(a,b) \not\geq 0 \quad \Leftrightarrow$ entanglement



Summary and Conclusions

Nonclassical correlations:

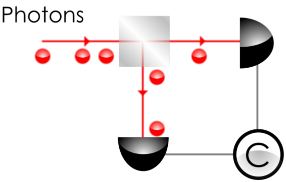
- Space-time dependent field correlations:

$$P(\{E^{(+)}(i)\}), \quad i \equiv (\mathbf{r}_i, t_i)$$

- Nonclassical correlations: $P \ngeq 0$

$\Rightarrow$ Conditions for general correlation functions

$$P(\{E^{(+)}(i)\}) \Rightarrow P_{\mathrm{Nfc}}(\{E^{(+)}(i)\})$$



Thanks to

Thomas Kiesel – P_{Ncl}

Jan Sperling – P_{Ent}

Soon available:

PhD position

Jan Sperling – P_{Ent}

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PhD position

Jan Sperling – P_{Ent}

Thank you for the attention!