Reconstruction of Quasiprobability Distributions Seminar

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Contents

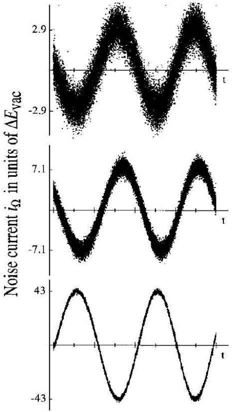
Coherent states as classical states

- Coherent states $|\alpha\rangle$ resemble classical oscillations
- Classical mixture of coherent states:

$$\hat{\rho}_{\mathrm{cl}} = \sum_i p_i |\alpha_i\rangle\langle\alpha_i| \Rightarrow \int d^2\alpha P_{\mathrm{cl}}(\alpha)|\alpha\rangle\langle\alpha|$$

$P_{\mathrm{cl}}(\alpha)$: classical probability density

- General quantum state: $\hat{\rho} = \int d^2\alpha\, P(\alpha)|\alpha\rangle\langle\alpha|$
- Nonclassical state: $P(\alpha) \cong$ quasi-probability, $P(\alpha) \neq P_{\mathrm{cl}}(\alpha)$



Coherent states as classical states

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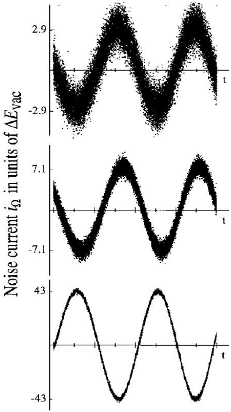


Consequences

- pure states are nonclassical if one requires quantum-mechanical superpositions of classical coherent states
- all statistical mixtures of coherent states are classical
- field correlation functions:

$$G_{m,n} = \langle \hat{E}^{(-)}(1) \dots \hat{E}^{(-)}(m) \hat{E}^{(+)}(n) \dots \hat{E}^{(+)}(1) \rangle$$

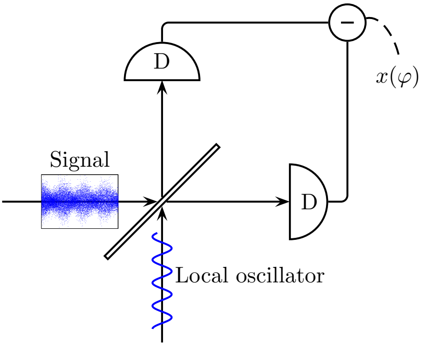
- can be measured in correlation experiments, e.g. HBT-setup:

$$G_{2,2} = \langle : \hat{I}(1) \hat{I}(2) : \rangle$$

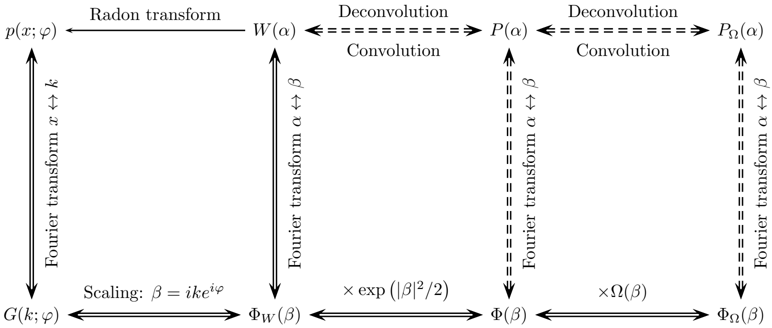
- can be explained classically if and only if P-representation is nonnegative

Balanced homodyne tomography

- overlap of signal with local oscillator at beamsplitter
- measurement of photocurrents
- difference current proportional to quadrature $x(\varphi)$
- phase φ fixed by relative phase of local oscillator
- result: set of N quadrature points $\{x_j(\varphi)\}_{j=1}^{N}$
- quadrature distributions $p(x;\varphi)$



Quasiprobabilities and characteristic functions



Data from balanced homodyne measurement: $\{x_j(\varphi)\}_{j=1}^{N}$

Exercise: Statistical uncertainty estimation

From measured quadratures $\{x_j(\varphi)\}_{j=1}^{N}$ to the characteristic function $G(k;\varphi)$:

$$G(k;\varphi) = \int p(x;\varphi)e^{ikx}dx \approx \frac{1}{N}\sum_{j=1}^{N} e^{ikx_j(\varphi)}.$$

1. Estimate the mean square deviation of this quantity,

$$\sigma^2\{G(k;\varphi)\} = \left\langle \left|G(k;\varphi)\right|^2\right\rangle - \left|\langle G(k;\varphi)\rangle\right|^2 .$$

   What happens in the limits $k \to 0$ and $|k| \to \infty$?

2. From $G(k;\varphi)$ to the filtered characteristic function of the P function:

$$\Phi_\Omega(\beta) = e^{|\beta|^2/2}G(|\beta|;\arg(\beta)-\pi/2)\Omega_w(\beta).$$

   Estimate the mean square deviation of $\Phi_\Omega(\beta)$ and discuss the behavior for large β.

3. Marginals of the filtered characteristic function of the P function:

$$p_\Omega(x;\varphi) = \int_{-\infty}^{\infty} e^{2ikx}e^{k^2/2}G(k;\varphi)\Omega_w(k)dk.$$

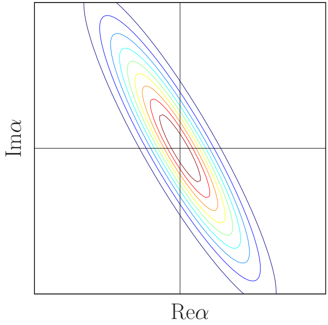
   Estimate the mean square deviation of this quantity.

Example: Phase-diffused squeezed vacuum states

- Wigner function:

$$W(\alpha) = \int f(\varphi) \frac{1}{2\pi\sqrt{V_x V_p}} \exp\left\{ -\frac{\mathrm{Re}^2(\alpha e^{-i\varphi})}{2V_x} - \frac{\mathrm{Im}^2(\alpha e^{-i\varphi})}{2V_p} \right\} d\varphi$$

- uncertainty relation: $V_x V_p \geq 1$
- Gaussian distribution $f(\varphi)$ with variance σ^2
- experiment[1] with $V_x = 0.36$, $V_p = 5.28$
- 10^7 quadrature points from balanced homodyne detection
- state is squeezed for $\sigma < 22.2°$



[1] Kiesel et. al., PRA **79**, 150505 (2009)

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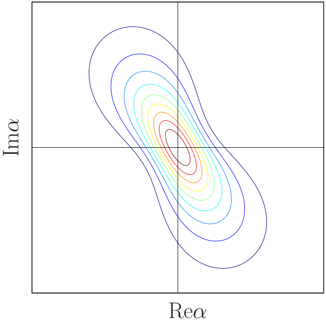


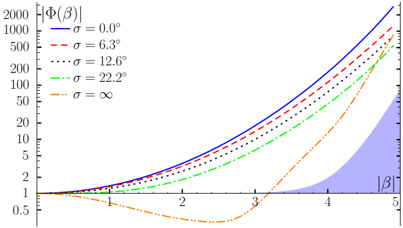
[1] Kiesel et. al., PRA **79**, 150505 (2009)

Example: Phase-diffused squeezed vacuum states

A state is nonclassical if[2]

$$\exists \beta \text{ with } |\Phi(\beta)| > 1$$

- all these states are nonclassical
- blue shaded area represents one standard deviation
- high significance of nonclassical effect
- characteristic function $\Phi(\beta)$ not integrable
- P function is highly singular



[2] Vogel, PRL **84**, 1849 (2000), Richter, Vogel, PRL **89**, 283601 (2002)

Nonclassicality filters

- problem: P is singular $\Leftrightarrow$ Φ is not integrable
- solution: filter characteristic function: $\Phi_\Omega(\beta) = \Phi(\beta)\Omega_w(\beta)$
- Requirements for the filter $\Omega_w(\beta)$ with width w:[3]
  - $\Phi_\Omega(\beta)$ is integrable $\Rightarrow$ regularized function $P_\Omega(\alpha)$
  - $P_{\Omega,\mathrm{cl}}(\alpha) \geq 0 \Rightarrow$ Fourier transform of $\Omega_w(\beta)$ is nonnegative
  - Original quantum state: $\lim_{w\to\infty} \Omega_w(\beta) = 1; \quad P_\Omega \Rightarrow P$
  - Suppression of statistical errors: $\Omega_w(\beta)e^{|\beta|^2/2}$ to be integrable
- **Such filters exist for arbitrary quantum states!**

[3] Kiesel and Vogel, PRA **82**, 032107 (2010)

Exercise: Autocorrelation filter

Let us define an autocorrelation filter $\Omega_w(\beta)$ as

$$\Omega_1(\beta) = \mathcal{N} \int \omega(\beta')\omega(\beta+\beta') d^2\beta', \quad \Omega_w(\beta) = \Omega_1(\beta/w),$$

with a normalization constant $\mathcal{N}$ chosen such that $\Omega_1(0) = 1$. Verify that

1. the Fourier transform $\int \Omega_1(\beta) e^{\alpha\beta^* - \alpha^*\beta} d^2\beta$ is nonnegative,
2. for $w \to \infty$, the filter approaches to one: $\lim_{w\to\infty} \Omega_w(\beta) = 1$,
3. if $\omega(\beta)$ is decreasing faster than any Gaussian function, in the sense that

$$\int \left| \omega(\beta) e^{a|\beta|^2} \right|^2 d^2\beta < \infty \qquad \forall a > 0,$$

then the same holds for $\Omega_1(\beta)$.

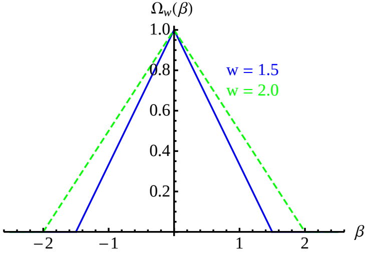
Hint: Show that if the latter inequality holds for some fixed a, then $|\Omega(\beta)| \leq C^2 e^{-a|\beta|^2/2}$ *with some suitable constant* C*. The Cauchy-Schwarz-inequality may be helpful:*

$$\left| \int f(\alpha) g(\alpha) d^2\alpha \right|^2 \leq \int |f(\alpha)|^2 d^2\alpha \int |g(\beta)|^2 d^2\beta$$

Example of onedimensional filters

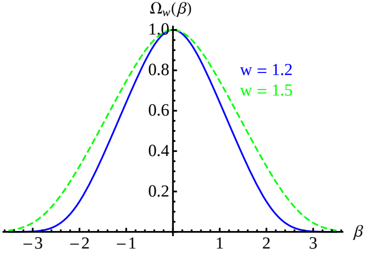
Triangular filter

$$\omega(\beta) = \begin{cases} 1 & |\beta| \leq 1/2, \\ 0 & \mathit{else} \end{cases}$$



Positive autocorrelation filter

$$\omega(\beta) = e^{-|\beta|^4}$$



Nonclassicality quasiprobabilities

- Additional requirement: $\Omega_w(\beta) \neq 0 \;\Rightarrow$ Filter is invertible
- Full information on the quantum state
- Regularized $P_\Omega \Rightarrow$ *nonclassicality quasiprobability:* P_Ω

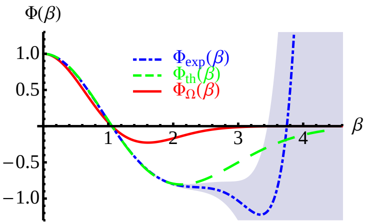
For any nonclassical quantum state, one can find negativities in $P_\Omega(\alpha)$ for sufficiently large filter width w.

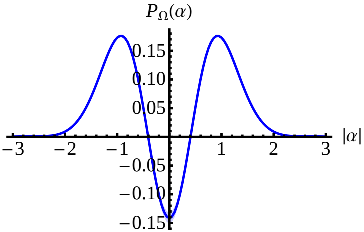
Scheme for a universal nonclassicality test

1. reconstruct characteristic function $\Phi(\beta)$ from balanced homodyne detection
2. choose filter $\Omega_w(\beta)$ and calculate $\Phi_\Omega(\beta) = \Phi(\beta)\Omega_w(\beta)$
3. calculate nonclassicality quasiprobability $P_\Omega(\alpha)$ by Fourier transform
4. increase filter width until negativities appear

Example: Single-photon-added thermal states

Photon added on thermal background ($\bar{n}_{\rm th} \approx 0.5$)[4]





Statistical significance of $P_{\Omega}(\alpha) < 0$: 15 standard deviations

[4]Kiesel, Vogel, Bellini, Zavatta, PRA **83**, 032119 (2011)

Exercise: Direct sampling of nonclassicality quasiprobabilities

The characteristic function $G(k;\varphi)$ is the Fourier transform of the quadrature distribution $p(x;\varphi)$,

$$G(k;\varphi) = \int_{-\infty}^{\infty} p(x;\varphi) e^{ikx} dx.$$

Furthermore, it is related to the nonclassicality quasiprobability $P_\Omega(\alpha)$ via

$$\begin{aligned} \Phi_\Omega(\beta) &= e^{|\beta|^2/2} G(|\beta|; \arg(\beta) - \pi/2) \Omega_w(\beta), \\ P_\Omega(\alpha) &= \frac{1}{\pi^2} \int d^2\beta \, \Phi_\Omega(\beta) e^{\alpha\beta^* - \alpha^*\beta}. \end{aligned}$$

Let us now examine an expression for $P_\Omega(\alpha)$ in the form

$$P_\Omega(\alpha) = \frac{1}{\pi} \int_0^{\pi} d\varphi \int_{-\infty}^{\infty} dx \, p(x;\varphi) f_\Omega(x,\varphi;\alpha,w). \quad (1)$$

1. Find a suitable function $f_\Omega(x;\varphi;\alpha,w)$.
2. What is the interpretation of Eq. (1)? How can it be practically implemented to experimental data? How can one estimate the resulting statistical error?

Example: Pattern function $f_\Omega(x,\varphi;\alpha,w)$

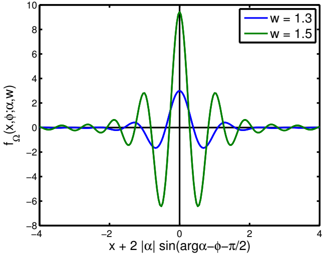
Pattern function for phase-independent filter:

$$f_\Omega(x,\varphi;\alpha,w) = \frac{1}{\pi}\int_{-\infty}^{\infty} db\,|b| e^{ib(x+2|\alpha|\sin(\arg\alpha-\varphi-\pi/2))} e^{b^2/2}\Omega_w(b)$$

Autocorrelation filter:

$$\Omega_1(\beta) = \mathcal{N}\int \omega(\beta')\omega(\beta+\beta')d^2\beta'$$

- here: $\omega(\beta) = \exp\{-|\beta|^4\}$
- the larger the width, the larger the oscillations



Reconstruction of a quasiprobability

Pattern function $f_\Omega(x, \varphi; \alpha, w)$:

$$P_\Omega(\alpha) = \langle f_\Omega(x, \varphi; \alpha, w) \rangle_{x,\varphi}$$

Estimate $P_\Omega(\alpha)$ as empirical mean of pattern function

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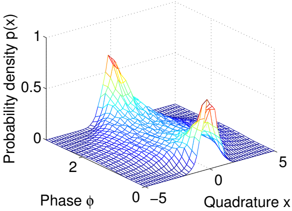

$x_1(\varphi_1)$

$\vdots$

$x_N(\varphi_N)$

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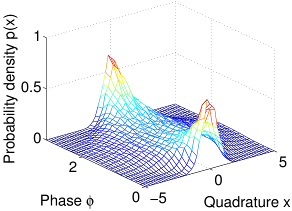
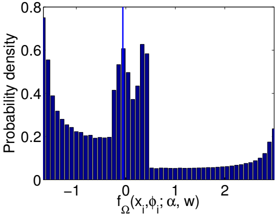
$$x_1(\varphi_1) \rightarrow f_1 = f_\Omega(x_1, \varphi_1; \alpha, w)$$

$$\vdots \qquad \vdots$$

$$x_N(\varphi_N) \rightarrow f_N = f_\Omega(x_N, \varphi_N; \alpha, w)$$





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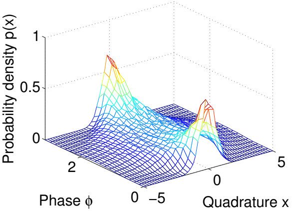
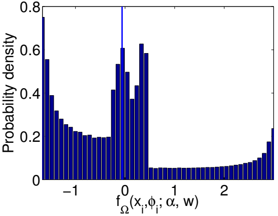
Estimate $P_\Omega(\alpha)$ as empirical mean of pattern function

$$x_1(\varphi_1) \quad \rightarrow \quad f_1 = f_\Omega(x_1, \varphi_1; \alpha, w)$$
$$\vdots \qquad\qquad \vdots$$
$$x_N(\varphi_N) \quad \rightarrow \quad f_N = f_\Omega(x_N, \varphi_N; \alpha, w)$$

Mean: $P_\Omega(\alpha) = \frac{1}{N}\sum_i f_\Omega(x_i, \varphi_i; \alpha, w)$
Variance: $\sigma^2\{P_\Omega\} = \sigma^2\{f_\Omega(x_i, \varphi_i; \alpha, w)\}$





Experimental result for squeezed state[5]

- squeezed vacuum with $V_x = 0.36\,V_{\rm vac}$ and $V_p = 5.28\,V_{\rm vac}$
- measurement: 10^5 quadratures for each of 21 phases
- significance of negativity optimized for $w = 1.3$
- quasiprobability shows clearly negativities
- these are not introduced by the filter
- standard deviation is less than linewidth
- signatures of nonclassicality

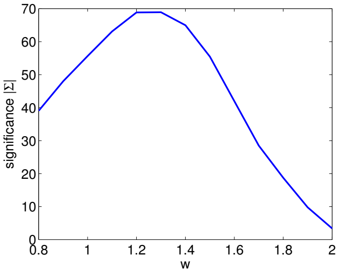


Figure: Significance of negativity in dependence of filter width w

[5] Kiesel, Vogel, Hage, Schnabel, arXiv:1103.2032 [quant-ph], PRL (in press).

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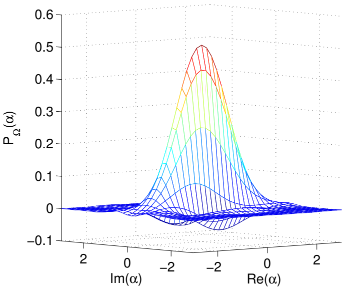

Figure: Nonclassicality quasiprobability

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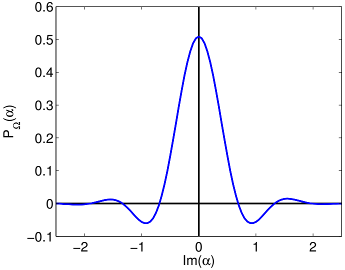

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Summary

- P representation: $\hat{\rho} = \int d^2\alpha P(\alpha)|\alpha\rangle\langle\alpha|$
- Nonclassical states do not have a nonnegative P function
- Problem: $P(\alpha)$ often highly singular
- Regularization: $P(\alpha) \Rightarrow P_\Omega(\alpha)$
- Negativities indicate nonclassicality of the state
- Applies to all quantum states; suppression of experimental noise
- Direct sampling from homodyne quadrature data possible

Thank you for your attention.